On Sequential Random Distortion Testing of Non-Stationary Processes[1]

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Outline

- Standard hypothesis testing vs Sequential testing
- Random Distortion Testing (RDT), Block-RDT
- **A new framework: SeqRDT**
  - Contributions
  - Model
  - Analysis
  - Simulations
- **Summary and future work**

Hypothesis Testing

To accept or reject the null hypothesis: $\mathcal{H}_0$ vs $\mathcal{H}_1$

- **Standard Testing:** Collect a fixed number of samples
  - Bayesian, Min-max or Neyman-Pearson hypothesis testing
  - Procedures are designed to minimize some cost
- **Sequential Testing** $\Rightarrow$ **Goal**: To make a decision faster on an average with sufficient performance guarantees
  1. Accept null hypothesis and stop the test
  2. Reject null hypothesis and stop the test
  3. Make no decision and collect another observation
- Continue taking observations unless a confident decision can be made

Sequential Testing: SPRT

- **When to stop the sequential test?**
- **Answer**: When the probability of false alarm and the probability of missed detection are under certain pre-specified levels (α and β resp.)

Standard Likelihood ratio test (LRT)

- Consider the two hypotheses as $\mathcal{H}_0$ and $\mathcal{H}_1$
- Consider the observations $\{Y_n\}$ for $n \in \mathbb{N}$
- The LRT yields

$$\Lambda_n(Y_1, \ldots, Y_n) = \log \frac{p_{Y_1,\ldots,Y_n}(Y_1, \ldots, Y_n; \mathrm{H}_1)}{p_{Y_1,\ldots,Y_n}(Y_1, \ldots, Y_n; \mathrm{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda$$

where, λ is the threshold chosen according to the type of test

Sequential Probability Ratio Test (SPRT)

SPRT has the following form:

- If $\Lambda_n \geqslant \lambda_H$, decide $\mathcal{H}_1$
- If $\Lambda_n \leqslant \lambda_L$, decide $\mathcal{H}_0$
- If $\lambda_L < \Lambda_n < \lambda_H$, take another observation to obtain Λ_{n+1}
- Repeat

The thresholds $\lambda_L = \log \frac{\beta}{1-\alpha}$ and $\lambda_H = \log \frac{1-\beta}{\alpha}$

- The probability of false alarm and missed detection stay under the specified levels α and β respectively
- SPRT is **optimal**

Random Distortion Testing (RDT)

- **Random Distortion Testing (RDT)**[2]
  - Binary hypothesis testing problem
  - Tests whether a signal lies within a ball of radius τ
- Block formulation of RDT: **Block-RDT**[3]
  - Tests whether the **signal mean** lies within a ball of radius τ

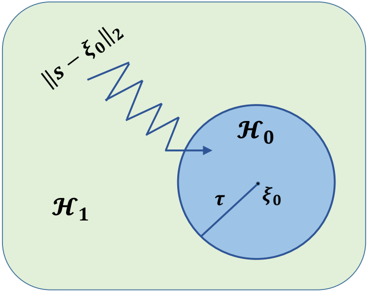


Figure 1: RDT.

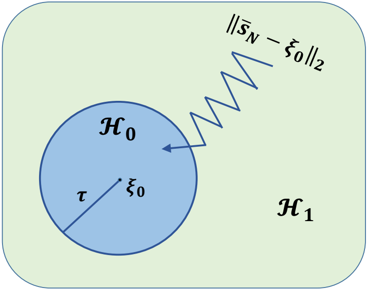


Figure 2: Block RDT.

[2] Pastor et al., IEEE TSP, 2013.
[3] Pastor et al., ICASSP, 2015.

Goal

- **Standard sequential testing approaches**
  - Based on LRT: Requires the knowledge of signal distributions
  - Assume the observation samples to be i.i.d.
  - Prone to model mismatch errors and hence are not robust
- **Advantages of RDT**:
  - The distributions of signal under each hypothesis is unknown
  - Samples need not be i.i.d.
  - Is robust to model mismatches
- **Question**:
  - Can we provide sufficient performance guarantees for the sequential testing framework?

Major Contributions

- **A new sequential testing framework**: *Seq*RDT
  - Makes a decision faster compared to Block RDT
  - Analysis of the proposed algorithm
  - Preliminary approaches[4,5]
- **Advantages**
  - Non-parametric sequential mean testing framework
  - Is capable of testing non-i.i.d. signals with unknown sample distributions
  - Robust to model mismatches

[4] Nguyen et al., PhD Dissertation, Telecom Bretagne, 2012

[5] Nguyen et al., BMC BioMedical Engineering OnLine, 2012.

Model

- **Idea:** Associate with each hypothesis a criterion
- Proposed in the form of Random distortion testing (RDT)[6] and *Block*RDT[7]

$$\begin{cases} \underline{\text{Observation}} : Y = S + X \\ \text{with} \quad \begin{cases} S = (S_n)_{n \in \mathbb{N}}, \\ X_1, X_2, \ldots \overset{\text{iid}}{\sim} \mathbb{F}, \ \mathbb{F} \text{ unknown.} \\ \mathbb{E}\left[\, X_1 \,\right] = 0 \text{ and } \operatorname{Var}\left(X_1\right) = 1 \\ S \text{ and } X \text{ are independent.} \end{cases} \\ \exists N_0 \in \mathbb{N}, \begin{cases} \mathcal{H}_0 : \forall N \geqslant N_0, 0 \leqslant |\bar{S}_N - \xi_0| \leqslant \tau \text{ (a-s)} \\ \mathcal{H}_1 : \forall N \geqslant N_0, \tau < |\bar{S}_N - \xi_0| \leqslant \tau_H \text{ (a-s)} \end{cases} \end{cases}$$

where, $\tau \in [0, \infty)$.

[6] Pastor et al., IEEE TSP, 2013.
[7] Pastor et al., ICASSP, 2015.

Model: Intuition

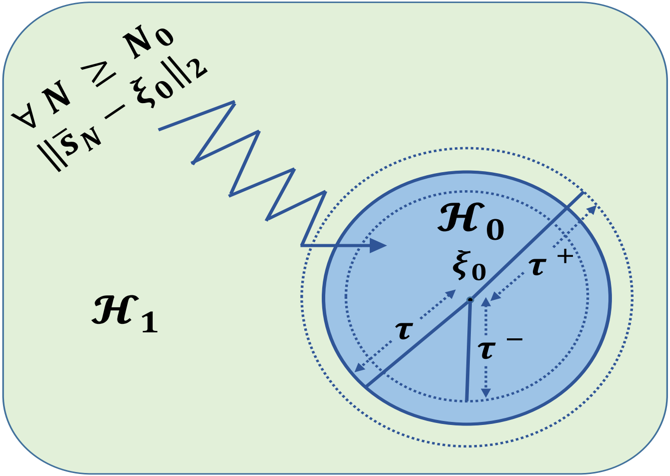


Figure 3: Testing model for *Seq*RDT

Assumption

- **Assumption 1: [(a-s) convergence of $\bar{S_N}$]**

$$\exists\, \bar{S_\infty} \text{ such that } \lim_{N\to\infty} \bar{S_N} = \bar{S_\infty} \ \text{(a-s)}$$

for which exist $\tau^- \in [0, \tau)$ and $\tau^+ \in (\tau, \infty)$ such that:

$$\begin{cases} \text{Under } \mathcal{H}_0: & |\bar{S_\infty} - \xi_0| \leqslant \tau^- \ \text{(a-s)}, \\ \text{Under } \mathcal{H}_1: & |\bar{S_\infty} - \xi_0| \geqslant \tau^+ \ \text{(a-s)}, \end{cases}$$

Remark: *Note that the two tolerances τ^- and τ^+ ensure that the non-parametric criterion $|\bar{S_N} - \xi_0|$ converges away from the tolerance τ as the number of samples N grows.*

SeqRDT

The stopping time T for the test is defined as

$$T = \min\left\{N \in \mathbb{N} : \mathcal{D}_M(N) \neq \infty\right\},$$

with:
$$\begin{cases} \mathcal{D}_M(1) = \mathcal{D}_M(2) = \ldots = \mathcal{D}_M(M) = \infty, \\ \text{for } N > M,\ \mathcal{D}_M(N) = \begin{cases} 0 & \text{if } |\bar{Y}_N - \xi_0| \leqslant \lambda_L(N), \\ \infty & \text{if } \lambda_L(N) < |\bar{Y}_N - \xi_0| \leqslant \lambda_H(N), \\ 1 & \text{if } |\bar{Y}_N - \xi_0| > \lambda_H(N). \end{cases} \end{cases}$$

where, $\lambda_L(N) \leqslant \lambda_H(N)$.

- **False Alarm:** $\mathbb{P}_{\mathrm{FA}}(\mathcal{D}_M) \stackrel{\text{def}}{=} \mathbb{P}\left[\mathcal{D}_M(T) = 1\right]$ under $\mathcal{H}_0$
- **Missed Detection:** $\mathbb{P}_{\mathrm{MD}}(\mathcal{D}_M) \stackrel{\text{def}}{=} \mathbb{P}\left[\mathcal{D}_M(T) = 0\right]$ under $\mathcal{H}_1$

Design thresholds such that: $\mathbb{P}_{\mathrm{FA}}(\mathcal{D}_M) \leqslant \alpha$ and $\mathbb{P}_{\mathrm{MD}}(\mathcal{D}_M) \leqslant \beta$

Thresholds

Choose:

$$\lambda_H(N) = \lambda_\alpha(\tau\sqrt{N})/\sqrt{N} \text{ and } \lambda_L(N) = \lambda_{1-\beta}(\tau\sqrt{N})/\sqrt{N}$$

where, given $\gamma \in (0,1)$ and $\rho \in [0,\infty)$, $\lambda_\gamma(\rho)$ satisfies

$$Q_{1/2}(\rho, \lambda_\gamma(\rho)) = \gamma.$$

Proposition

For $\alpha, \beta \in (0, 1/2)$, $\tau \in (0,\infty)$ and the thresholds $\lambda_H(N)$ and $\lambda_L(N)$ we have, $\lambda_L(N) \leqslant \lambda_H(N)$, for all $N \in \mathbb{N}$.

- Can this choice of thresholds give some performance guarantees?

Behaviour of $\lambda_L(N)$ and $\lambda_H(N)$

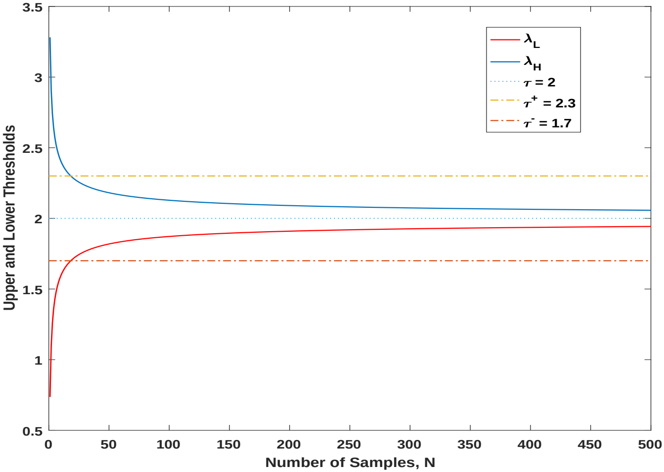


Figure 4: $\lambda_L(N)$ and $\lambda_H(N)$ vs N for $\alpha = \beta = 0.1$ and $\tau = 2$.

Theorem

Theorem (Asymptotics: T, $\mathbb{P}_{\mathrm{FA}}(\mathcal{D}_M)$ and $\mathbb{P}_{\mathrm{MD}}(\mathcal{D}_M)$)

For $\alpha, \beta \in \left(0, \frac{1}{2}\right)$ and thresholds $\lambda_L(N)$ and $\lambda_H(N)$,

(i) we have

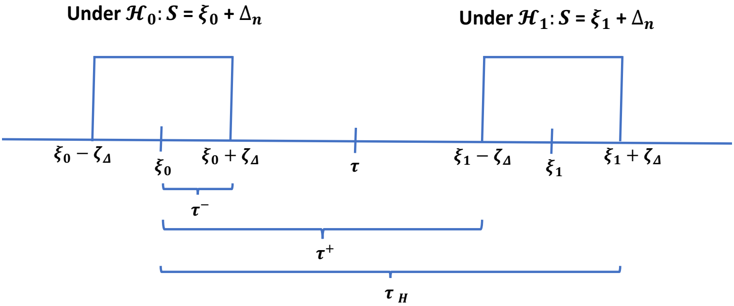
$$\mathbb{P}\,[\,T = \infty\,] = 0 \text{ under } \mathcal{H}_0 \text{ and } \mathcal{H}_1,$$

(ii) and as $M \to \infty$, we have

$$\lim_{M\to\infty} \mathbb{P}_{\mathrm{FA}}(\mathcal{D}_M) = \lim_{M\to\infty} \mathbb{P}_{\mathrm{MD}}(\mathcal{D}_M) = 0$$

Experimental Results and Discussion

- Signal: $S_n = \xi_i + \Delta_n$ under $\mathcal{H}_i$ for $i \in \{0, 1\}$
- Δ_ns: Distortions with unknown distribution



The above inequalities imply the following

$$\begin{cases} \text{under } \mathcal{H}_0: \ 0 \leqslant |\bar{S}_N - \xi_0| \leqslant \tau^- < \tau, \forall N \geqslant 1, \\ \text{under } \mathcal{H}_1: \ \tau < \tau^+ \leqslant |\bar{S}_N - \xi_0| \leqslant \tau_H, \forall N \geqslant 1. \end{cases}$$

Simulations

$\alpha = \beta = 0.001$					
SDR $\left(= \frac{\text{SNR}}{\tau}\right)$ (dB)		6.02	7.96	9.54	12.00
Seq-RDT, $M = 0$	T	3.98	3.28	3.04	2.90
	$\mathbb{P}_{\mathrm{FA}}(\mathcal{D}_M)$	3.33×10^{-4}	3.47×10^{-4}	3.19×10^{-4}	3.17×10^{-4}
	$\mathbb{P}_{\mathrm{MD}}(\mathcal{D}_M)$	1.24×10^{-4}	5×10^{-6}	5×10^{-6}	0
*Block*RDT	$N_{\text{B-RDT}}$	14	7	4	2
	$P_{\text{FA}}^{\text{B-RDT}}$	0	0	0	2.2×10^{-5}
	$P_{\text{MD}}^{\text{B-RDT}}$	9.45×10^{-4}	3.12×10^{-4}	2.44×10^{-4}	6.80×10^{-5}
SPRT	T_{SPRT}	2.44	1.73	1.34	1.05
	$P_{\text{FA}}^{\text{SPRT}}$	2.08×10^{-4}	1.59×10^{-4}	1.03×10^{-4}	2.85×10^{-5}
	$P_{\text{MD}}^{\text{SPRT}}$	2.09×10^{-4}	1.54×10^{-4}	1.03×10^{-4}	2.57×10^{-5}
SPRT-MM	$T_{\text{SPRT-MM}}$	1.57	1.24	1.10	1.01
	$P_{\text{FA}}^{\text{SPRT-MM}}$	6.2×10^{-3}	3.5×10^{-3}	1.8×10^{-3}	2.88×10^{-4}
	$P_{\text{MD}}^{\text{SPRT-MM}}$	6.2×10^{-3}	3.6×10^{-3}	1.8×10^{-3}	3.05×10^{-4}

Table 1: *Seq*-RDT vs *Block*RDT and SPRT for unbounded regime.

Summary: SeqRDT

- A robust sequential algorithm for hypothesis testing
- Underlying distributions are unknown
- No i.i.d assumptions are made over the signals
- **Ongoing and Future Work**
  - Truncated version of *Seq*RDT: T-*Seq*RDT
  - Distributed version of *Seq*RDT: D-*Seq*RDT
  - Generalization to higher dimensional signals