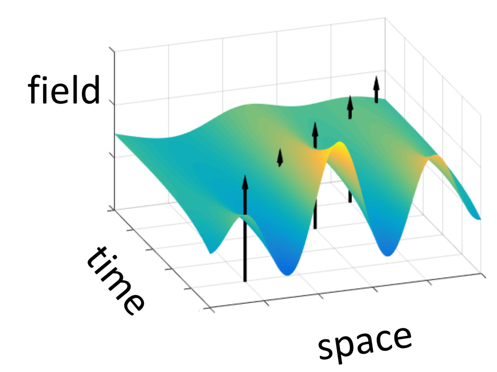
Bandlimited Spatiotemporal Field Sampling with Location and Time Unaware Sensors

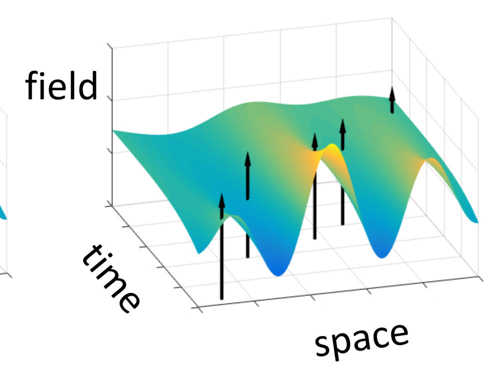
Sudeep Salgia and Animesh Kumar
Indian Institute of Technology Bombay
Mumbai, India

Introduction

Consider a sampling problem, where a spatio-temporal field governed by a linear constant coefficient partial differential equation (PDE) is sampled by a mobile sensor. Contrast the following:

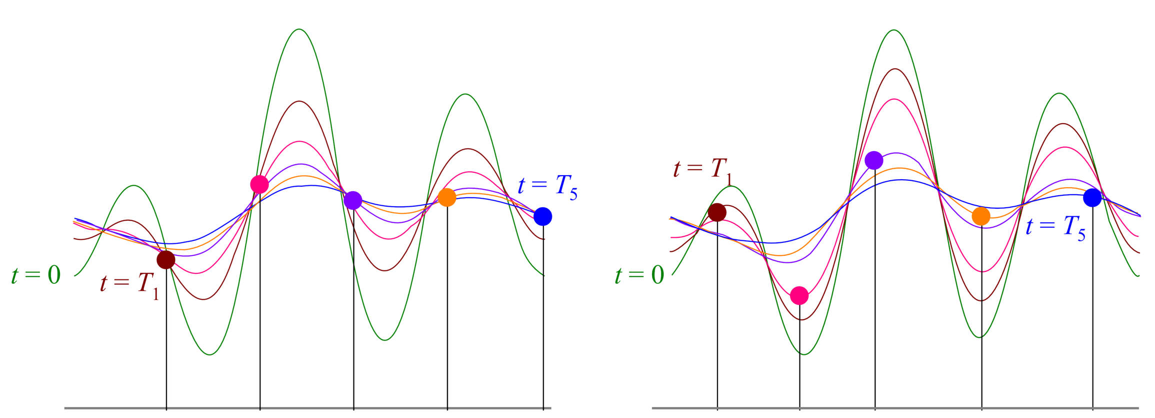


classical: uniform sampling with known timestamps



our work: both the location and timestamps of mobile sensor are unknown

The sampling process can be depicted via the following plots:



Motivation

	Economic	A location-unaware and time-unaware mobile sensor will avoid the costs of GPS, other accurate localization mechanisms, and a precise clock
	Social	A location-unaware sensor will preserve the privacy of the mobile sensor (assuming it is with a social device)
	Academic	What is the fundamental impact of not knowing the sample locations in spatial field reconstruction problems?

Analytical setup

Spatial field model

A finite support field that is spatially bandlimited field with bandwidth b

$$g(x,t) = \sum_{k=-b}^{b} a_k(t)\exp(j2\pi kx)$$

Field evolves according to a known constant coefficient linear PDE, such as the heat equation

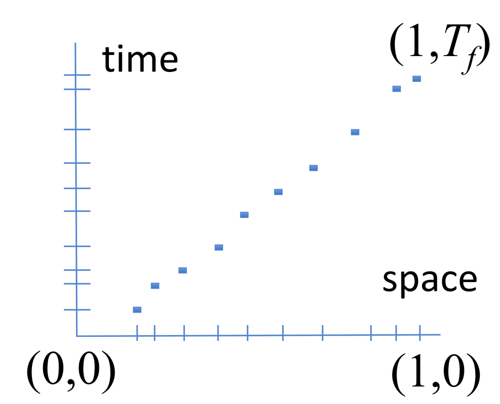
$$\frac{\partial}{\partial t} g(x,t) = c\frac{\partial^2}{\partial x^2} g(x,t)$$

or more generally by

$$\sum_{i=0}^{m} p_i \frac{\partial^i}{\partial t^i} g(x,t) = \sum_{i=0}^{m'} q_i \frac{\partial^i}{\partial x^i} g(x,t)$$

Analytical setup

Sampling location and sampling time model



Sampling points form two independent unknown renewal processes in space and in time

The mean of M is the average sampling density, i.e., n

In details
- sensing starts at $x = 0$, $t = 0$ ends before $x = 1$, $t = T_f$
- the intersample distances are a realization of an unknown renewal process
- the intersample times are a realization of another independent unknown renewal process

Distortion criterion

For any estimate of the field

$$\mathcal{D}\left[\hat{G}, g\right] = \mathbb{E}\left[\int_0^1 |\hat{G}(x,t) - g(x,t)|^2\right]\Bigg|_{t=0} = \mathbb{E}\left[\sum_{k=-b}^{b} |\hat{A}_k(t) - a_k(t)|^2\right]\Bigg|_{t=0}$$

Noise model

It is assumed that each measured sample $g(x, t)$ is affected by an independent and identically distributed, zero-mean, finite variance noise $W(x, t)$

Our inference algorithm

- $g(x, t)$ can be written as an inner product, of Fourier basis dependent vectors and location dependent coefficients
$$g(s_i, t_i) = \mathbf{e}(s_i)^H \mathbf{a}(t_i)$$
- Using linearity in the Fourier coefficients, the problem is cast as a linear regression to estimate the Fourier coefficients at $t = 0$
$$\hat{A}_k(0) = \arg\min_b ||\mathbf{g} - Y\mathbf{b}||_2^2$$
- where $\mathbf{g}$ is a vector formed by measurement-noise affected samples, Y is a matrix formed by the location dependent vectors and $\mathbf{b}$ is the vector of target Fourier coefficients
- For regression, the (location-time) pair of samples are approximated as
$$T_i \approx iT_f/M \text{ and } S_i \approx i/M$$

Main Result

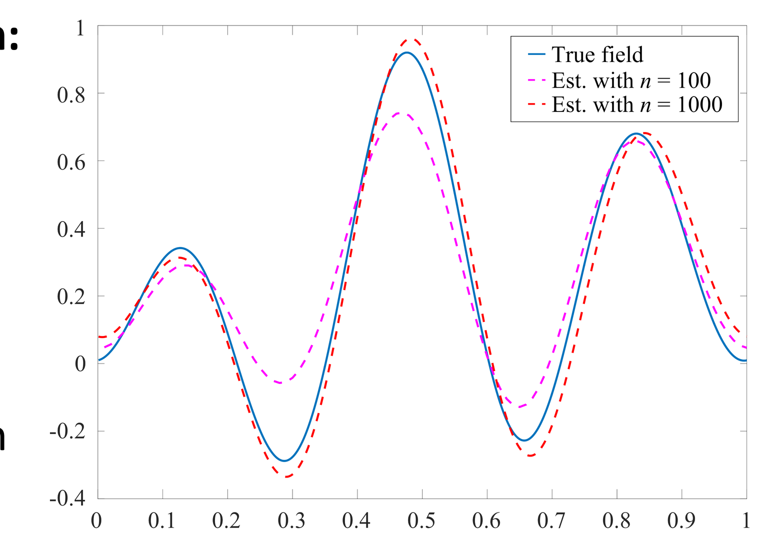
Theorem: Let $\hat{A}_k(0), -b \le k \le b$ be the output of our inteference algorithm. Then, the mean-squared error and therefore the distortion D is bounded as

$$\sum_{k=-b}^{b} \mathbb{E}[|\hat{A}_k(0) - a_k(0)|^2] \le O(1/n)$$

The key result of our paper as well as its challenge is an analytical proof of the above

Theorem's illustration: The field's bandwidth is $b = 3$, and its coefficients at $t = 0$ are generated from uniformly distributed random variables. The diffusion equation was used as PDE:

$$\frac{\partial}{\partial t} g(x,t) = 0.01\frac{\partial^2}{\partial x^2} g(x,t)$$



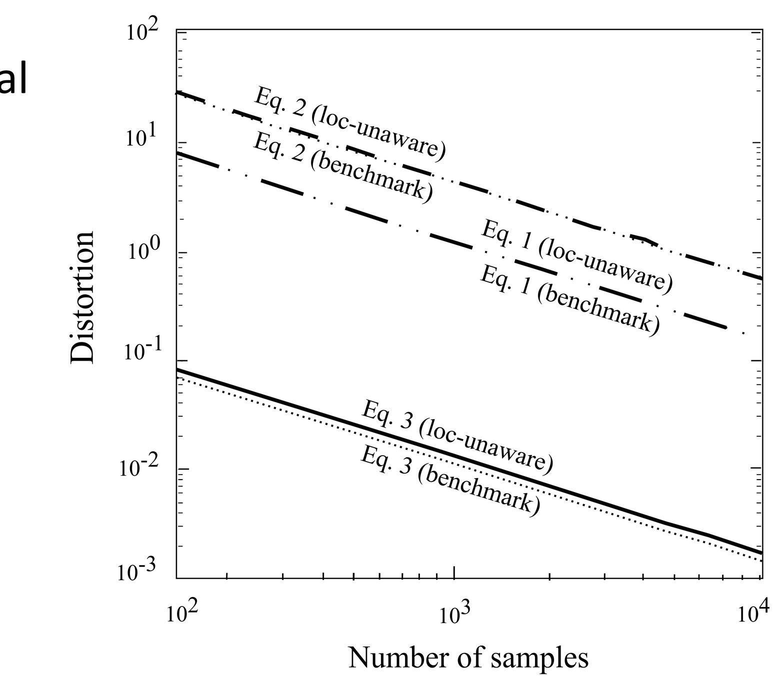
Related works

Mobile sensing:
- Unnikrishnan and Vetterli'2013 (the idea of using a mobile sensor and associated aliasing/path density tradeoffs)
- Kumar'2017 (location-unaware mobile sensing with temporally fixed fields)

Ongoing developments in location-unaware sensing:
- Kumar'2015 (scattered location-unaware sensors and associated results)
- Pacholska, Haro, Scholefield, Vetterli'2017 (uniqueness constraints for ensuring field reconstruction)
- SLAM algorithm and its variants

Simulations

Three different PDEs were examined:
- $g(x, 0)$ was generated by uniformly distributed Fourier series coefficients. The final field was scaled to ensure $|g(x, 0)| \le 1$
- The initial field evolved with the following PDEs:
  - Eq. 1, $\mathbf{p} = (2, 3, 0)$, $\mathbf{q} = (-0.000125, 0, 0.01, 0, 0)$
  - Eq. 2, $\mathbf{p} = (1, 3, 0)$, $\mathbf{q} = (0.01, 0, 0)$
  - Eq. 3, $\mathbf{p} = (1, 0)$, $\mathbf{q} = (0.01, 0, 0)$
- A benchmark was also used where locations of samples were random but known
- The renewal processes were generated using uniformly distributed random variables



Conclusions and future work

- Spatiotemporal and initially bandlimited fields evolving by a linear PDE can be reconstructed from location-unaware samples taken on an unknown renewal process
- The mean-squared error scales as $O(1/n)$, where n is the average number of samples
- The regression framework is universal, since neither it requires the renewal process distributions nor the noise distribution
- Exploration of two-dimensional spatial fields is of interest
- Exploration of approximately bandlimited field's sampling and reconstruction is of interest