Locally Optimal Invariant Detector for Testing Equality of Two Power Spectral Densities

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Goal

Determine whether two **multivariate** time series possess the same (matrix-valued) power spectral density at **every** frequency

Applications

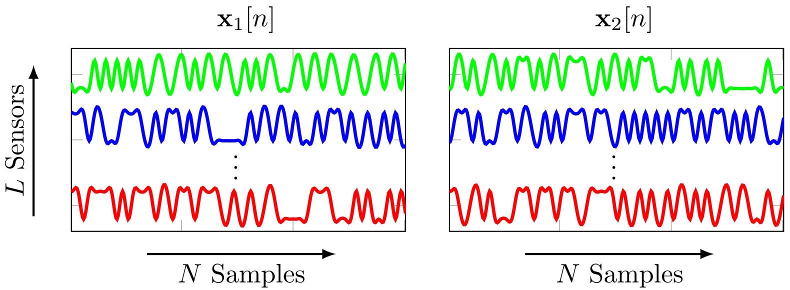
- Spectrum sensing
- Physical-layer security
- Comparison of gas pipes
- Earthquake-explosion discrimination
- ...

Previously proposed detectors

- Frequency-domain detectors
  - Non-parametric detectors based on periodograms
  - Generalized likelihood ratio test (GLRT)
  - Extensions of tests for homogeneity of covariance matrices
- Time-domain detectors

Main objective

Study the existence of optimal detectors



Stack of N observations

$$\mathbf{y}_i = \left[\mathbf{x}_i^T[0] \quad \cdots \quad \mathbf{x}_i^T[N-1]\right]^T$$

Hypothesis test

Defining $\mathbf{y} = [\mathbf{y}_1^T \ \mathbf{y}_2^T]^T$ and assuming Gaussianity

$$\mathcal{H}_1 : \mathbf{y} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_{\mathcal{H}_1})$$
$$\mathcal{H}_0 : \mathbf{y} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_{\mathcal{H}_0})$$

Covariance matrices under both hypotheses

$$\mathbf{R}_{\mathcal{H}_1} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix} \qquad \mathbf{R}_{\mathcal{H}_0} = \begin{bmatrix} \mathbf{R}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_0 \end{bmatrix}$$

where, under $\mathcal{H}_1$, $\mathbf{R}_1 \neq \mathbf{R}_2$ and $\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}_0$ under $\mathcal{H}_0$

Covariance matrices of each process

$$\mathbf{R}_i = E[\mathbf{y}_i \mathbf{y}_i^H] = \begin{bmatrix} \mathbf{M}_i[0] & \cdots & \mathbf{M}_i[N-1] \\ \vdots & \ddots & \vdots \\ \mathbf{M}_i[N-1] & \cdots & \mathbf{M}_i[0] \end{bmatrix}$$

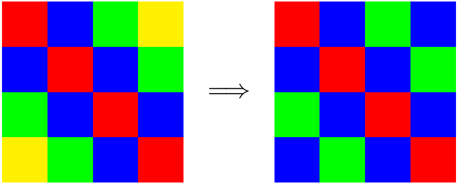
where $\mathbf{M}_i[m] = E[\mathbf{x}_i[n]\mathbf{x}_i^H[n-m]]$

Block-Toeplitz structure

- Not enough invariances for deriving optimal detectors

Asymptotic case

- Asymptotically approximate block-Toeplitz matrices by block-circulant ones: MSE convergence of the likelihoods



Block-circulant matrices

$$\mathbf{R}_i \approx (\mathbf{F}_N \otimes \mathbf{I}_L)\mathbf{S}_i(\mathbf{F}_N \otimes \mathbf{I}_L)^H$$

Fourier matrix ($\mathbf{F}_N$)

block-diagonal with blocks ($\mathbf{S}_i$)

$$\mathbf{S}_i(e^{j\theta_k}) = \sum_{n=0}^{N-1} \mathbf{M}_i[n] e^{-j\theta_k n}$$

Transformation of the observations

Transform the observations as $\mathbf{z} = [\mathbf{z}_1^T \ \mathbf{z}_2^T]^T$ with

$$\mathbf{z}_i = (\mathbf{F}_N \otimes \mathbf{I}_L)^H \mathbf{y}_i$$

Asymptotic likelihood of z

$$\log p(\mathbf{z}^{(0)}, \ldots, \mathbf{z}^{(M-1)}; \mathbf{S}_{\mathcal{H}_i}) \propto -\log\det(\mathbf{S}_{\mathcal{H}_i}) - \operatorname{tr}\left(\mathbf{S}_{\mathcal{H}_i}^{-1}\hat{\mathbf{S}}\right)$$

where

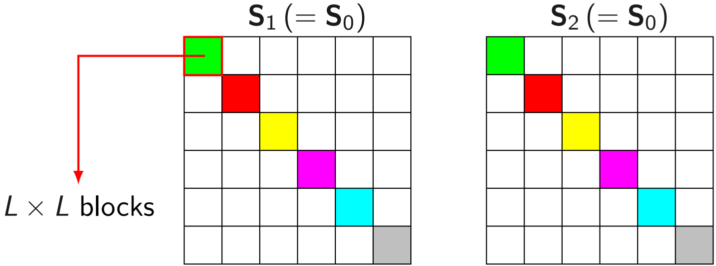
$$\hat{\mathbf{S}} = \frac{1}{M}\sum_{m=0}^{M-1} \mathbf{z}^{(m)}\mathbf{z}^{(m)H}$$

and

$$\mathbf{S}_{\mathcal{H}_1} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix} \qquad \mathbf{S}_{\mathcal{H}_0} = \begin{bmatrix} \mathbf{S}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_0 \end{bmatrix}$$

Covariance matrices in the frequency domain

- $\mathbf{S}_i$ is block-diagonal with block size L



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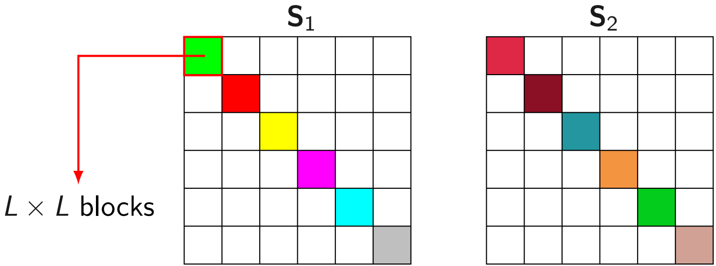


Typical approach for deriving the UMPIT (or LMPIT)

- Problem invariances and transformation group
- Maximal invariant statistic
- Densities of the maximal invariant statistic
- Ratio of the densities of the maximal invariant statistic
- (Local approximation of the ratio)

Wijsman's Theorem

$$\mathscr{L} = \frac{\int_{\mathcal{G}} p\left(g(\mathbf{x}); \mathcal{H}_1\right) |\det(\mathbf{J}_g)| dg}{\int_{\mathcal{G}} p\left(g(\mathbf{x}); \mathcal{H}_0\right) |\det(\mathbf{J}_g)| dg}$$

Maximal invariant statistic and its distributions not required

Problem invariances

- Multiply $\mathbf{S}_i$ by a block-diagonal matrix (block size L)
- Reorder the frequencies (for both processes)
- Relabel the processes on a frequency-by-frequency basis

Invariant group

$$\mathcal{G} = \left\{ g : \mathbf{z} \to g(\mathbf{z}) = \tilde{\mathbf{G}}\mathbf{z} \right\}$$

with

$$\tilde{\mathbf{G}} = (\mathbf{I}_2 \otimes \mathbf{G}) \left(\prod_{k=1}^{N} \mathbf{P}_k \otimes \mathbf{I}_L \right) (\mathbf{I}_2 \otimes \mathbf{T} \otimes \mathbf{I}_L)$$

Ratio of the distributions of the maximal invariant

$$\mathscr{L} = \frac{\displaystyle\sum_{\mathbb{T},\mathbb{P}_1,\ldots,\mathbb{P}_N} \int_{\mathbb{G}^N} |\det(\mathbf{G})|^{4M} \exp\left\{-M\mathrm{tr}\left(\mathbf{S}_{\mathcal{H}_1}^{-1}\tilde{\mathbf{G}}\hat{\mathbf{S}}\tilde{\mathbf{G}}^H\right)\right\} d\mathbf{G}}{\displaystyle\sum_{\mathbb{T},\mathbb{P}_1,\ldots,\mathbb{P}_N} \int_{\mathbb{G}^N} |\det(\mathbf{G})|^{4M} \exp\left\{-M\mathrm{tr}\left(\mathbf{S}_{\mathcal{H}_0}^{-1}\tilde{\mathbf{G}}\hat{\mathbf{S}}\tilde{\mathbf{G}}^H\right)\right\} d\mathbf{G}}$$

after a lot of algebra and a Taylor series

LMPIT?

$$\mathscr{L} \propto \sum_{i=1}^{2}\sum_{k=1}^{N} \|\hat{\mathbf{C}}_{i,k}\|^2 + \alpha \sum_{i=1}^{2}\sum_{k=1}^{N} \mathrm{tr}^2(\hat{\mathbf{C}}_{i,k})$$

where

$$\hat{\mathbf{C}}_{i,k} = \left[\frac{1}{2}\left(\hat{\mathbf{S}}_{1,k} + \hat{\mathbf{S}}_{2,k}\right)\right]^{-1/2} \hat{\mathbf{S}}_{i,k} \left[\frac{1}{2}\left(\hat{\mathbf{S}}_{1,k} + \hat{\mathbf{S}}_{2,k}\right)\right]^{-1/2}$$

LMPIT-inspired detectors

$$\mathscr{L}_1 = \sum_{i=1}^{2} \sum_{k=0}^{N-1} \|\hat{\mathbf{C}}_i(e^{j\theta_k})\|_F^2 \qquad \mathscr{L}_2 = \sum_{i=1}^{2} \sum_{k=1}^{N} \operatorname{tr}^2(\hat{\mathbf{C}}_i(e^{j\theta_k}))$$

where

$$\hat{\mathbf{C}}_i(e^{j\theta}) = \left[\frac{1}{2}\sum_{i=1}^{2}\hat{\mathbf{S}}_i(e^{j\theta})\right]^{-1/2} \hat{\mathbf{S}}_i(e^{j\theta}) \left[\frac{1}{2}\sum_{i=1}^{2}\hat{\mathbf{S}}_i(e^{j\theta})\right]^{-1/2}$$

LMPIT for univariate time series ($L = 1$)

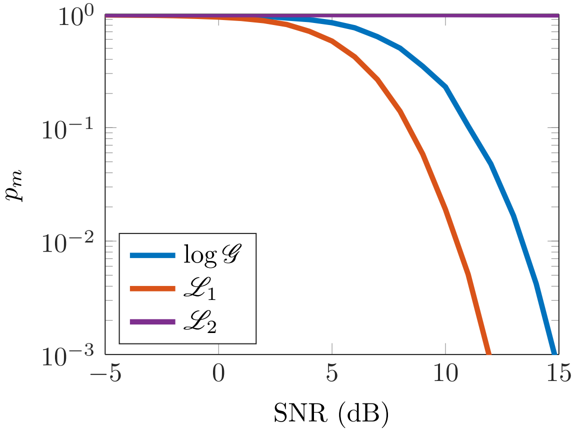
$$\mathscr{L} \propto \sum_{k=0}^{N-1} \frac{\hat{S}_1^2(e^{j\theta_k}) + \hat{S}_2^2(e^{j\theta_k})}{\left[\frac{1}{2}\left(\hat{S}_1(e^{j\theta_k}) + \hat{S}_2(e^{j\theta_k})\right)\right]^2}$$

System model

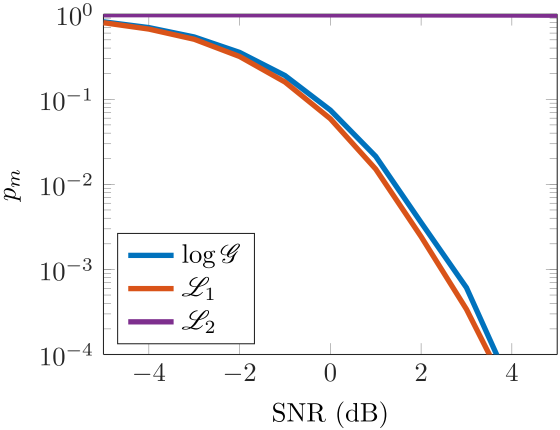
$$\mathbf{x}_i[n] = \sum_{\tau=0}^{T-1} \mathbf{H}_i[\tau]\mathbf{s}_i[n-\tau] + \mathbf{v}_i[n], \quad i = 1, 2$$

where

- $\mathbf{s}_i[n] \in \mathbb{C}^Q$ are independent QPSK signals
- $\mathbf{v}_i[n] \in \mathbb{C}^L$ are independent white noises
- $\mathbf{H}_1[n]$ is a Rayleigh MIMO channel with unit energy, spatially uncorrelated, and exponential pdp with parameter ρ
- $\mathbf{H}_2[n] = \sqrt{1-\Delta_h}\mathbf{H}_1[n] + \sqrt{\Delta_h}\mathbf{E}[n]$



$\Delta_h = 0.1$ under $\mathcal{H}_1$ and $\Delta_h = 0$ under $\mathcal{H}_0$. The number of transmitted signals is $Q = 1$



$\Delta_h = 0.1$ under $\mathcal{H}_1$ and $\Delta_h = 0$ under $\mathcal{H}_0$. The number of transmitted signals is $Q = 5$

- We have proved the non-existence of the UMPIT/LMPIT for testing whether two multivariate signals posses the same PSD
- For univariate time series we have derived the LMPIT
- We have proposed two LMPIT-inspired detectors