The Asynchronous Power Iteration: A Graph Signal Perspective

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California Institute of Technology

43rd International Conference on
Acoustics, Speech and Signal Processing

Caltech

Outline

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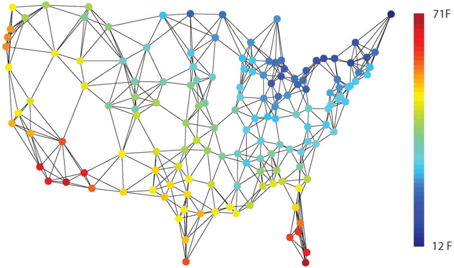
1. Graph Signal Processing
2. Autonomous Networks and Graph Signals
   - Asynchronous Updates
   - Convergence Results
   - Asynchronicity and Smoothness
   - Distributed Computation of the Graph Eigenvectors
3. Conclusion

Outline

Preliminaries



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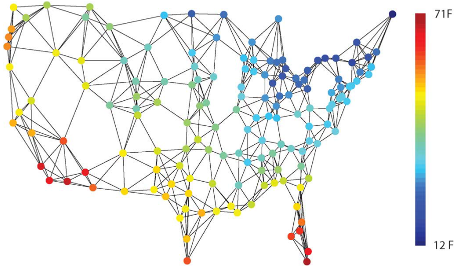

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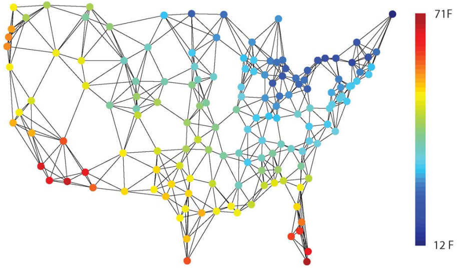

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Graph Laplacians[2]	:	$\boldsymbol{L}$, or $\mathcal{L}$
Other selections[3]		

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[2] Shuman et al, "The emerging field of signal processing on graphs: ...," *IEEE S. P. Magazine, vol. 30, no. 3 2013*

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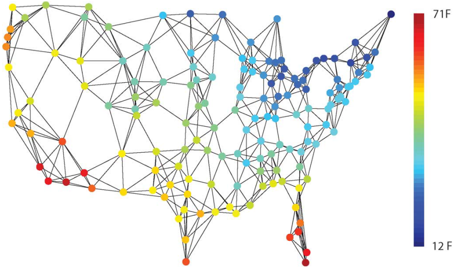

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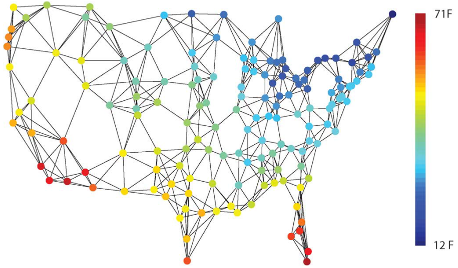

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Graph Fourier Transform : $\boldsymbol{F} = \boldsymbol{V}^{-1}$

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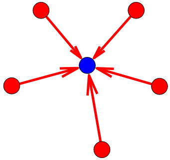

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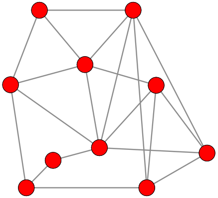

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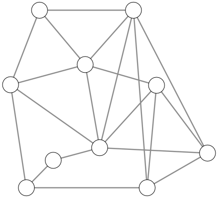

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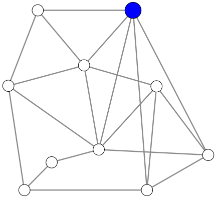

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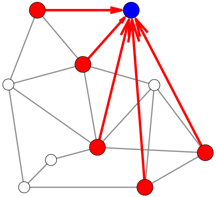

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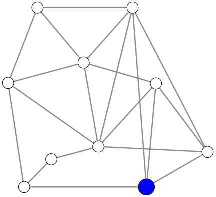

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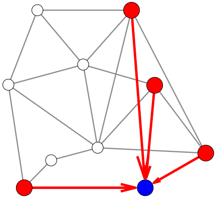

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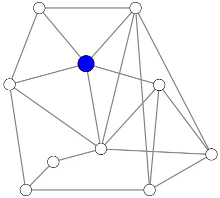

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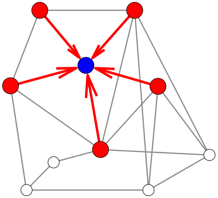

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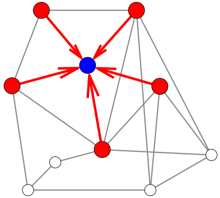

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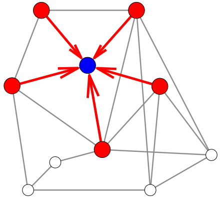

Recurrent NN
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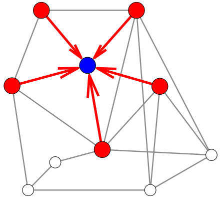

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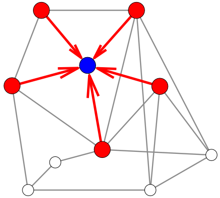

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Synchronous case:

$\lambda = 1, \qquad |\lambda| < 1$

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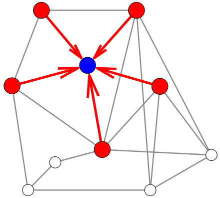

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Corollary

Synchronous $\Longrightarrow$ *Asynchronous*

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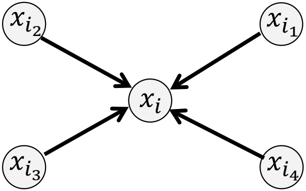

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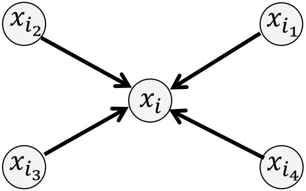

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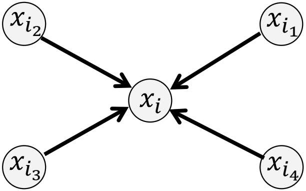

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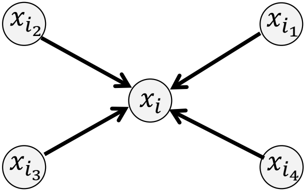

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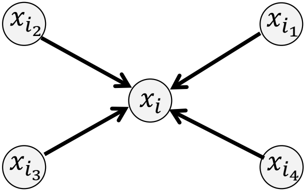

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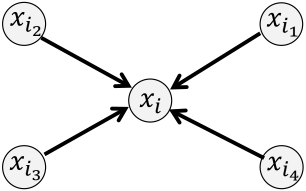

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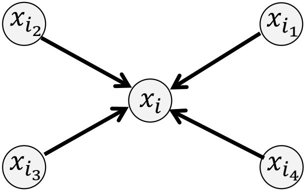

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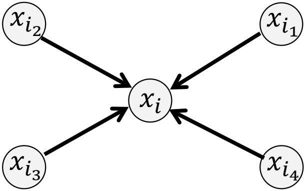

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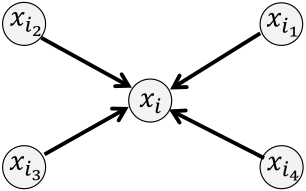

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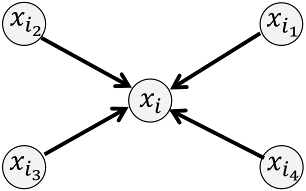

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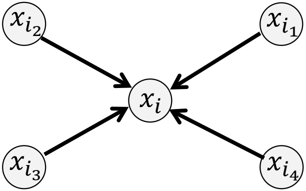
↙ Sync. ↘ Fully Async.

Amount of Asynchronicity

The Form of the Asynchronous Updates

$$x_k[i] = \begin{cases} \boldsymbol{a}_i \, \boldsymbol{x}_{k\text{-}1}, & i \in \mathcal{T}, \\ x_{k\text{-}1}[i], & i \notin \mathcal{T}. \end{cases}$$



$\mathcal{T}$ = *random subset*

↙ Content ↘ Size = t

Equally likely among $\binom{N}{t}$

$$\mathbb{P}(\,\mathcal{T}\,) = \binom{N}{t}^{-1}$$

$$1 \leqslant t \leqslant N$$

$$\delta_T = \frac{N-t}{N-1}$$

$$0 \leqslant \delta_T \leqslant 1$$

↙ Sync. ↘ Fully Async.

Amount of Asynchronicity

[1] Teke & Vaidyanathan, "Random Asynchronous Updates on Graphs," *IEEE TSP, submitted*

Expectation of Fourier Coefficients

$\boldsymbol{x}_0$

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$\boldsymbol{x}_0 \rightarrow \boldsymbol{x}_1 \rightarrow \cdots \rightarrow \boldsymbol{x}_k \rightarrow \cdots$

Expectation of Fourier Coefficients

$$\color{blue}{\boldsymbol{x}_0} \color{red}{\rightarrow} \boldsymbol{x}_1 \color{red}{\rightarrow} \cdots \color{red}{\rightarrow} \boldsymbol{x}_k \color{red}{\rightarrow} \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

Update on Random Subsets

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$$\boldsymbol{A} = \boldsymbol{V}\,\boldsymbol{\Lambda}\,\boldsymbol{V}^*$$

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$$\mathbb{E}[\widehat{x}_{k,\,j}] = \left(1 + \frac{t}{N}\,(\lambda_j - 1)\right)^k \widehat{x}_{0,\,j}$$

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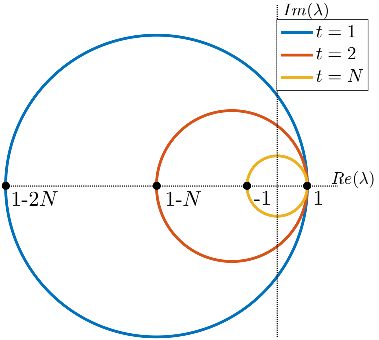

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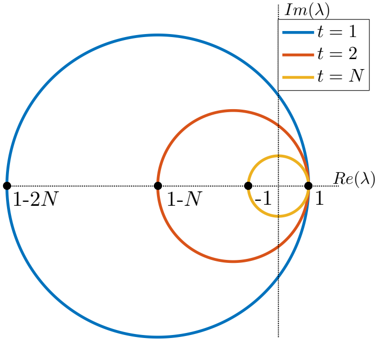

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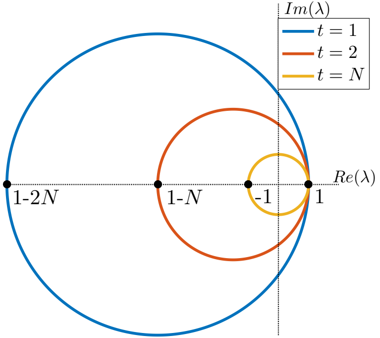

Convergence in Mean-Squared

[1] Teke & Vaidyanathan, "Random Asynchronous Updates on Graphs," *IEEE TSP, submitted*

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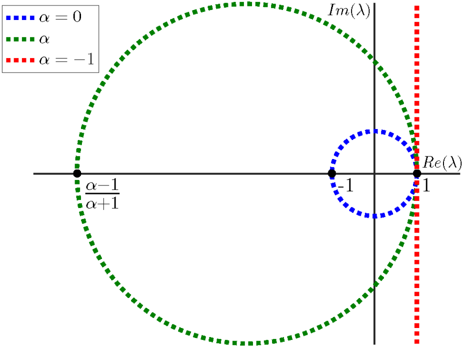

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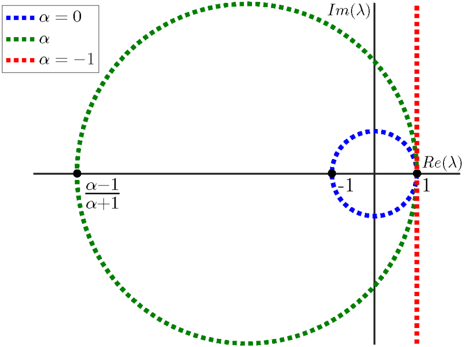

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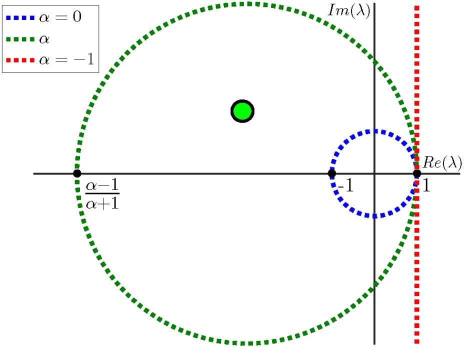

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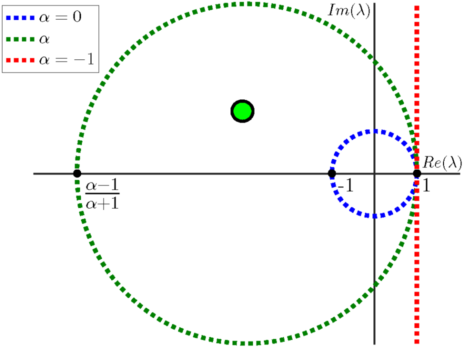

[1] Percha et al.,"Transition from local to global phase synchrony in small world NN and its possible impl. for epilepsy," *Phys.Rev.E*, *2005*
[2] Uhlhaas & Singer, "Neural synchrony in brain disorders: ... ," *Neuron*, *2006*

A Toy Example

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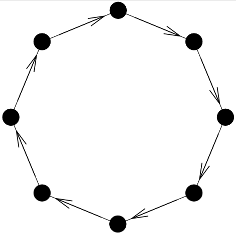

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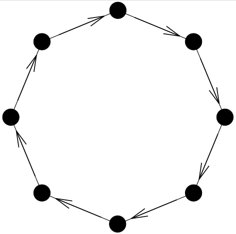

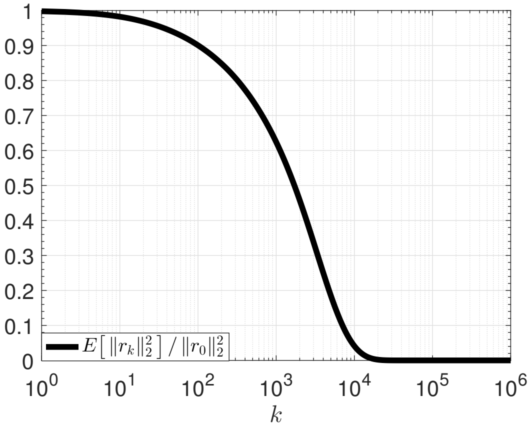

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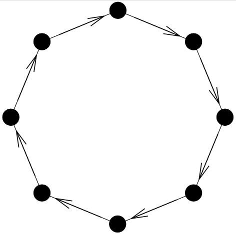

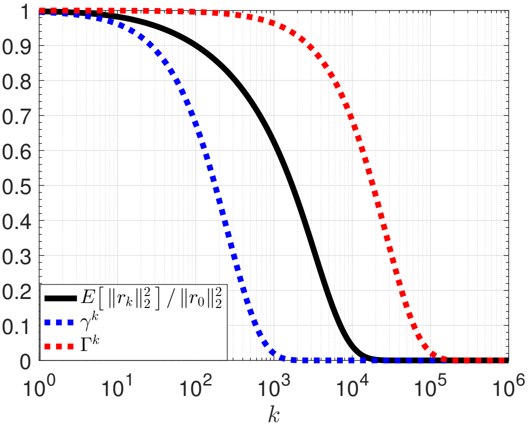

$$\gamma = 1 - \frac{4}{N^2}, \qquad \Gamma = 1 - \frac{4\pi^2}{N^4}$$

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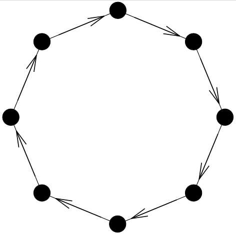

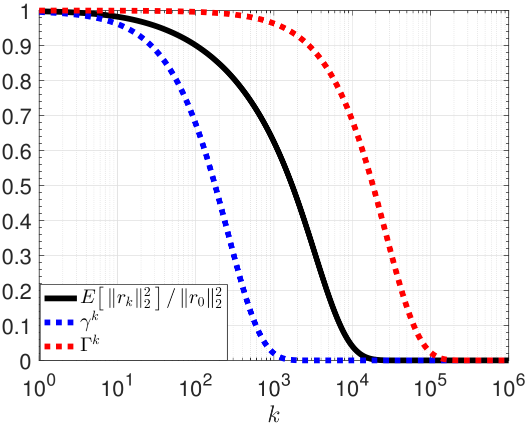

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Interpretations in terms of Smoothness $(A = V\Lambda V^*)$

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Definition (Smoothness Set)

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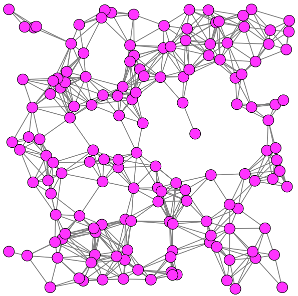

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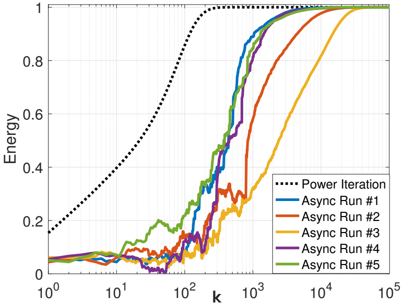

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Polynomial Filtering

$\boldsymbol{A}$

Asynchronous

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$\lambda = 1$	$\boldsymbol{A}$
$\|\lambda(\boldsymbol{A})\| < 1$	Asynchronous

Polynomial Filtering

$$\begin{matrix} \lambda = 1 \\ |\lambda(\boldsymbol{A})| < 1 \end{matrix} \quad \xrightarrow[\text{Asynchronous}]{\boldsymbol{A}} \quad \lim_{k \to \infty} \boldsymbol{x}_k \in \text{null}(\boldsymbol{A} - \boldsymbol{I})$$

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$$H(\boldsymbol{A}) = \sum_{k=0}^{L} h_k\, \boldsymbol{A}^k$$

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- Trade-offs (*Complexity* v.s. Rate) (*Spectral Information* v.s. Rate)

[1] Teke & Vaidyanathan, "Random Asynchronous Updates on Graphs," *IEEE TSP, submitted*

Outline

Conclusions & Future Work

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- Random update model
- Convergence behavior
  - Eigenspace geometry
  - Amount of asynchronicity

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Thank you!