ON THE ROLE OF THE BOUNDED LEMMA IN THE SDP FORMULATION OF ATOMIC NORM PROBLEMS

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What is this study about?

- Atomic norm is a useful abstraction **[1]**
- Tractable SDP formulations exist for line spectra **[2]**
- State-Space Models & DTBL **[3]**
- *Equivalence between two existential results* **[4]**

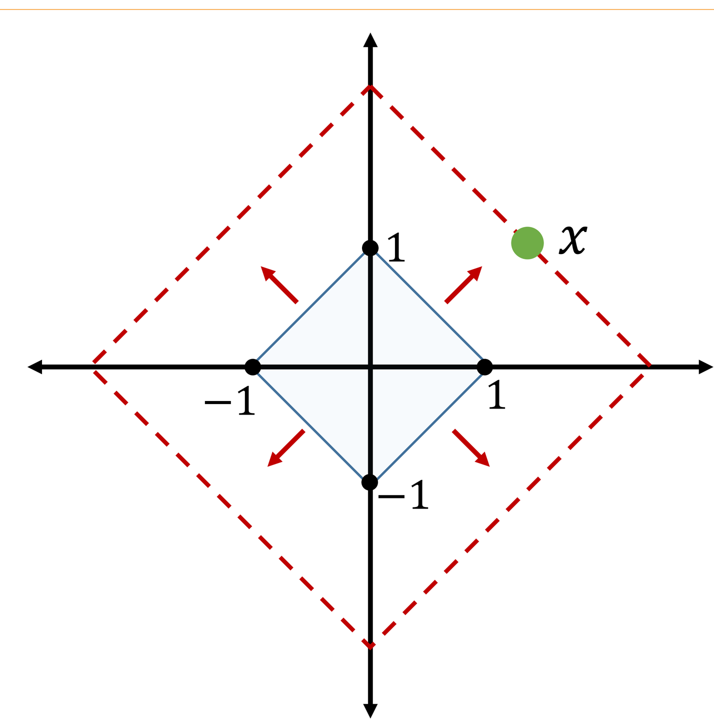
Atomic Norm

$\mathcal{A} \Rightarrow$ Dictionary $\quad \|\boldsymbol{x}\|_{\mathcal{A}} = \inf\left\{ t > 0 \mid \boldsymbol{x} \in t \cdot conv(\mathcal{A}) \right\}$

$\mathcal{A} = [\,\boldsymbol{I} \quad -\boldsymbol{I}\,]$

$conv(\mathcal{A}) = \ell_1 - ball$

$\|\boldsymbol{x}\|_{\mathcal{A}} = \|\boldsymbol{x}\|_1$



$\mathcal{A}$ = Rank-1 matrices $\Longrightarrow$ $\|\cdot\|_{\mathcal{A}} = \|\cdot\|_*$

$\mathcal{A}$ = Orthogonal matrices $\Longrightarrow$ $\|\cdot\|_{\mathcal{A}} = \|\cdot\|_2$

$$\boldsymbol{x} = \sum_i c_i\, \boldsymbol{a}_i \quad \Longrightarrow \quad \|\boldsymbol{x}\|_{\mathcal{A}} = \min_{c_i} \sum_i |c_i|$$

Atomic Norm Minimization Problems

$$\boldsymbol{y} = \boldsymbol{\Phi}\,\boldsymbol{x} \qquad \min_{\boldsymbol{x}} \ \|\boldsymbol{x}\|_{\mathcal{A}} \quad s.t. \quad \boldsymbol{y} = \boldsymbol{\Phi}\,\boldsymbol{x}$$

$$\Updownarrow$$

$$\max_{\boldsymbol{z}} \ \langle \boldsymbol{y}, \boldsymbol{z} \rangle \quad s.t. \quad \|\boldsymbol{\Phi}^* \boldsymbol{z}\|_{\mathcal{A}}^* \leq 1$$

$$\|\boldsymbol{x}\|_{\mathcal{A}}^* = \sup_{\boldsymbol{a} \in \mathcal{A}} \langle \boldsymbol{x}, \boldsymbol{a} \rangle$$

Toeplitz Operator and Its Adjoint

$$T(\boldsymbol{\mu}) = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_N \\ \mu_2^* & \mu_1 & \cdots & \mu_{N\text{-}1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_N^* & \mu_{N\text{-}1}^* & \cdots & \mu_1 \end{bmatrix}, \qquad T^{\dagger}(\boldsymbol{Q}) = \begin{bmatrix} \vdots \\ \sum_{i=1}^{N\text{-}j} Q_{i,\,i+j} \\ \vdots \end{bmatrix}$$

Line Spectrum Estimation

$$x_n = \sum_{k=1}^{K} c_k\, e^{j\,\pi\, n\, f_k}, \qquad n = 0, \cdots, N\text{-}1, \qquad c_k \in \mathcal{C}$$

Caratheodary method

$$\underbrace{\begin{bmatrix} x(K\text{-}1) & x(K\text{-}2) & \cdots & x(0) \\ x(K) & x(K\text{-}1) & \cdots & x(1) \\ \cdots & & \cdots & \cdots \\ x(2K\text{-}1) & x(2K\text{-}2) & \cdots & x(K) \end{bmatrix}}_{\boldsymbol{X} \in \mathcal{C}^{(K+1)\times K}}$$

$$\boldsymbol{u}^*\, \boldsymbol{X} = \boldsymbol{0}$$

$$U(f) = \sum_{k=0}^{K} u(k)\, e^{j\,\pi\, f\, k} = 0$$

MUSIC

$$\boldsymbol{S}_{xx} = \mathbb{E}\left[\boldsymbol{x}\,\boldsymbol{x}^*\right]$$

$$\boldsymbol{S}_{xx} = [\,\boldsymbol{V}_s \ \ \boldsymbol{V}_n\,]\, \boldsymbol{\Lambda}\, [\,\boldsymbol{V}_s \ \boldsymbol{V}_n\,]^*$$

$$\boldsymbol{w}(f) = \left[1 \ \ e^{j\,\pi\, f} \ \cdots \ e^{j\,\pi\,(N\text{-}1)\,f}\right]^T$$

$$P(f) = \frac{1}{\|\, \boldsymbol{V}_n^* \ \ \boldsymbol{w}(f)\,\|_2^2}$$

Atomic Dictionary for Line Spectrum

$$\boldsymbol{a}(f, \phi) = e^{j\pi\phi}\left[\,1 \ \ e^{j\,\pi\, f} \ \ e^{j\,\pi\,2\,f} \cdots e^{j\,\pi\,(N\text{-}1)\,f}\,\right]^T \in \mathcal{C}^{N\times 1}$$

$$\mathcal{A} = \left\{ \boldsymbol{a}(f, \phi) \;\middle|\; f \in [\text{-}1, 1), \ \phi \in [0, 2) \right\}.$$

$$\|\boldsymbol{x}\|_{\mathcal{A}}^* = \sup_{f,\ \phi} \ e^{-j\pi\phi} \underbrace{\sum_{n=0}^{N\text{-}1} x_n\, e^{-j\,\pi\, n\, f}}_{X(f)} = \sup_{f} \ |X(f)|$$

$$\|\boldsymbol{x}\|_{\mathcal{A}}^* \leq \tau \quad \Longleftrightarrow \quad |X(f)| \leq \tau \quad \forall f.$$

Non-Negative Trigonometric Polynomials

$X(f)$ is a polynomial in $e^{-j\,\pi\, f}$

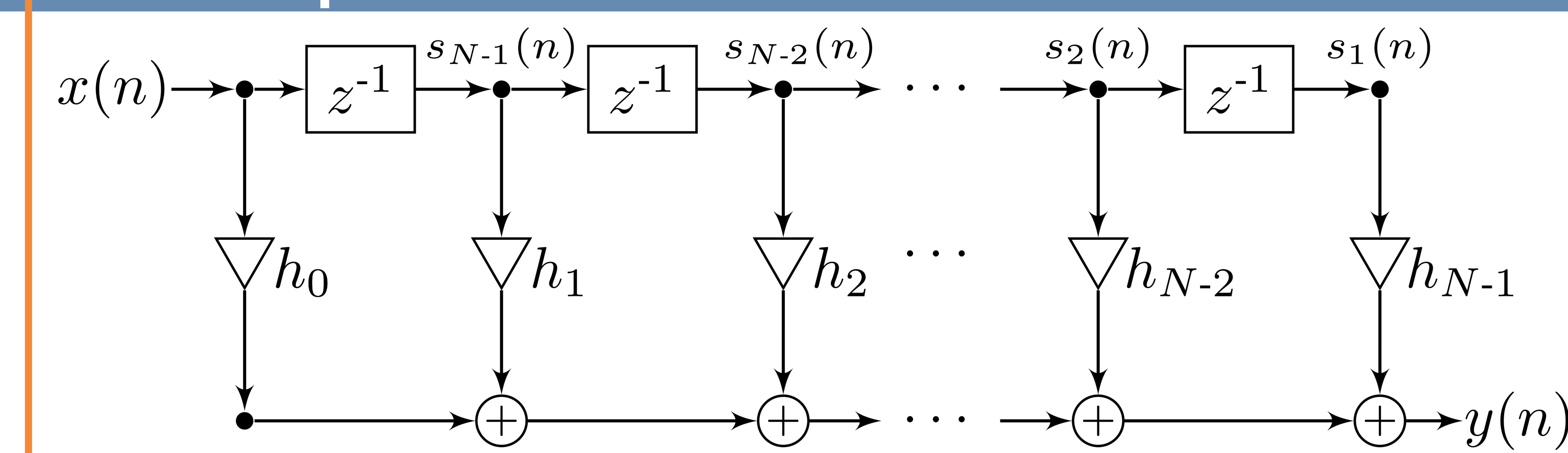
$$|X(f)| \leq \tau \iff \exists\, \boldsymbol{Q} \ \text{s.t.} \ \sum_{i=1}^{N\text{-}j} Q_{i,\,i+j} = \tau^2 \delta(j), \quad \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{x} \\ \boldsymbol{x}^* & 1 \end{bmatrix} \succeq \boldsymbol{0}$$

Atomic Norm Problems ($\boldsymbol{\Phi} = \boldsymbol{I}$)

$$\max_{\boldsymbol{z},\ \boldsymbol{Q} \in \mathcal{S}^N} \ \langle \boldsymbol{y}, \boldsymbol{z} \rangle \quad \text{s.t.} \quad \sum_{i=1}^{N\text{-}j} Q_{i,\,i+j} = \delta(j), \quad \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{z} \\ \boldsymbol{z}^* & 1 \end{bmatrix} \succeq \boldsymbol{0}$$

$$\min_{\boldsymbol{x},\, \boldsymbol{\mu},\, t} \ \frac{\mu_1 + t}{2} \quad \text{s.t.} \quad \begin{bmatrix} T(\boldsymbol{\mu}) & \boldsymbol{x} \\ \boldsymbol{x}^* & t \end{bmatrix} \succeq \boldsymbol{0}$$

State-Space Models



$$\boldsymbol{s}(n{+}1) = \boldsymbol{A}\,\boldsymbol{s}(n) + \boldsymbol{B}\,x(n)$$
$$y(n) = \boldsymbol{C}\,\boldsymbol{s}(n) + \boldsymbol{D}\,x(n)$$

$$\begin{bmatrix} \boldsymbol{s}(n{+}1) \\ y(n) \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} \boldsymbol{s}(n) \\ x(n) \end{bmatrix}, \qquad \boldsymbol{R} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \in \mathcal{C}^{N\times N},$$

$$\widehat{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\,\boldsymbol{T}, \qquad \widehat{\boldsymbol{B}} = \boldsymbol{T}^{-1}\boldsymbol{B}, \qquad \widehat{\boldsymbol{C}} = \boldsymbol{C}\,\boldsymbol{T}, \qquad \widehat{\boldsymbol{D}} = \boldsymbol{D}$$

Discrete-Time Bounded Lemma

For an FIR filter $\boldsymbol{h} = [h_{N\text{-}1} \ \ h_{N\text{-}2} \ \cdots \ h_0]$

$$|H(f)| \leq 1 \quad \Longleftrightarrow \quad \exists\, \boldsymbol{R} \ \ \text{s.t.} \ \ \boldsymbol{R}^*\, \boldsymbol{R} \preceq \boldsymbol{I}$$

★ Main Result ★

For an FIR filter $\boldsymbol{h} = [h_{N\text{-}1} \ \ h_{N\text{-}2} \ \cdots \ h_0]$ with $h_{N\text{-}1} \neq 0$

$$\exists\, \boldsymbol{R} \ \ \text{s.t.} \ \ \boldsymbol{R}^*\, \boldsymbol{R} \preceq \boldsymbol{I}$$

$$\Updownarrow$$

$$\exists\, \boldsymbol{Q} \in \mathcal{S}^N \ \ \text{s.t.} \ \ \sum_{i=1}^{N\text{-}j} Q_{i,\,i+j} = \delta(j), \quad \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{h}^* \\ \boldsymbol{h} & 1 \end{bmatrix} \succeq \boldsymbol{0}$$

★ *SDP formulation follows readily from well-known system theory*
★ *Proof depends only on linear algebra and it is self-contained*
★ *Makes the connections more transparent*

References

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