The Network Nullspace Property for Compressed Sensing of Big Data over Networks

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Background

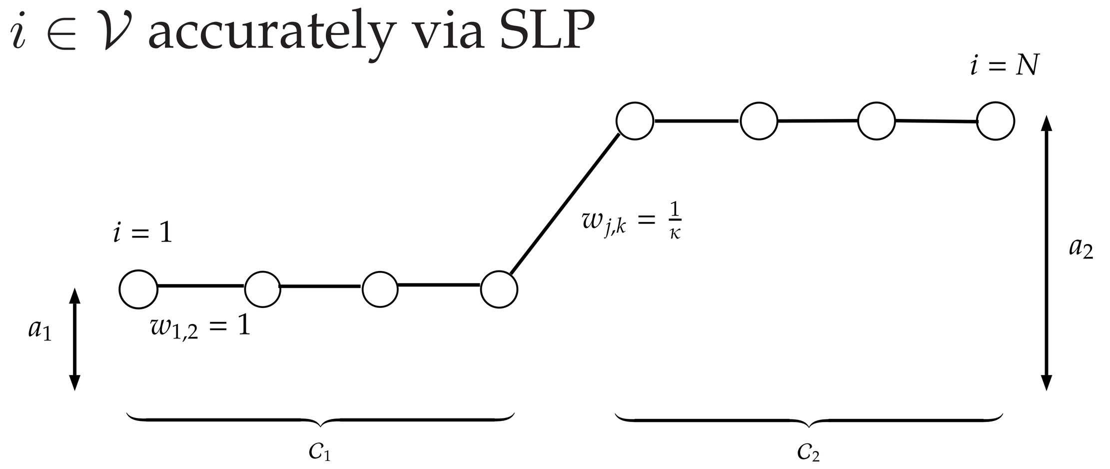
- Sparse Label Propagation (SLP) [1] extends Label Propagation by requiring signal differences over edges to be sparse
- SLP takes the network structure of data into account
- SLP learns entire graph signals from few samples
- SLP amounts to a convex optimization method based on the primal-dual method popularized by Chambolle and Pock [2]

Contribution

- we introduce the Network Nullspace Property (NNSP)
- NNSP involves the sampling set and the cluster structure of the underlying graph
- NNSP requires the existence of network flows with demands
- NNSP is a sufficient condition for accurate recovery of clustered graph signals via SLP

Graph Signal Recovery

- given:
  - data graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$
  - nodes represent data points, edges correspond to similarities (correlations) between data points
  - weighted adjacency matrix $\mathbf{W} \in \mathbb{R}_{+}^{N \times N}$
  - signal samples $\mathbf{y}[i] = \mathbf{x}[i] + \varepsilon[i]$, for $i \in \mathcal{M}$
  - sampling set $\mathcal{M}$ is small compared to the size of graph $|\mathcal{M}| \ll |\mathcal{V}|$
- goal:
  - recover the clustered graph signal value $\mathbf{x}[i]$ for all nodes $i \in \mathcal{V}$ accurately via SLP



Sparse Label Propagation

- clustered graph signals have small total variation:

$$\|\mathbf{x}\|_{TV} := \sum_{\{i,j\} \in \mathcal{E}} \mathbf{W}_{i,j} |\mathbf{x}[j] - \mathbf{x}[i]|$$

- SLP amounts to the convex optimization problem:

$$\hat{\mathbf{x}} \in \arg\min_{\tilde{\mathbf{x}} \in \mathbb{R}^{\mathcal{V}}} \|\tilde{\mathbf{x}}\|_{TV} \text{ s.t. } \tilde{\mathbf{x}}[i] = \mathbf{x}[i] \text{ for all } i \in \mathcal{M} \quad (1)$$

Piece-wise Constant Graph Signals

- we consider piece-wise constant (clustered) graph signal

$$\mathbf{x}[i] = \sum_{\mathcal{C} \in \mathcal{F}} \mathbf{a}_{\mathcal{C}} \mathcal{I}_{\mathcal{C}}[i], \quad (2)$$

  with some partition $\mathcal{F} = \{\mathcal{C}_1, \ldots, \mathcal{C}_{|\mathcal{F}|}\}$.
- NOTE: SLP does not require knowledge of $\mathcal{F}$!

Network Flows with Demands [3]

Definition 1 *A flow with demands $g[i] \in \mathbb{R}^{\mathcal{V}}$, for $i \in \mathcal{V}$, is a mapping $f[\cdot] : \mathcal{E} \to \mathbb{R}$ satisfying the conservation law:*

$$\sum_{j \in \mathcal{N}^{+}(i)} f[\{i,j\}] - \sum_{j \in \mathcal{N}^{-}(i)} f[\{i,j\}] = g[i]$$

at every node $i \in \mathcal{V}$.

Network Nullspace Property (NNSP)

Definition 2 *A sampling set $\mathcal{M} \subseteq \mathcal{V}$ is said to satisfy the NNSP, if for any signature $\sigma_e \in \{-1, +1\}^{\partial \mathcal{F}}$, which assigns the sign σ_e to a boundary edge $e \in \partial \mathcal{F}$, there is a flow $f[e]$ with demands $g[i] = 0$ for $i \notin \mathcal{M}$, and*

$$f[e] = 2\sigma_e W_e \text{ for } e \in \partial \mathcal{F}, f[e] \leq W_e \text{ for } e \in \mathcal{E} \backslash \partial \mathcal{F}.$$

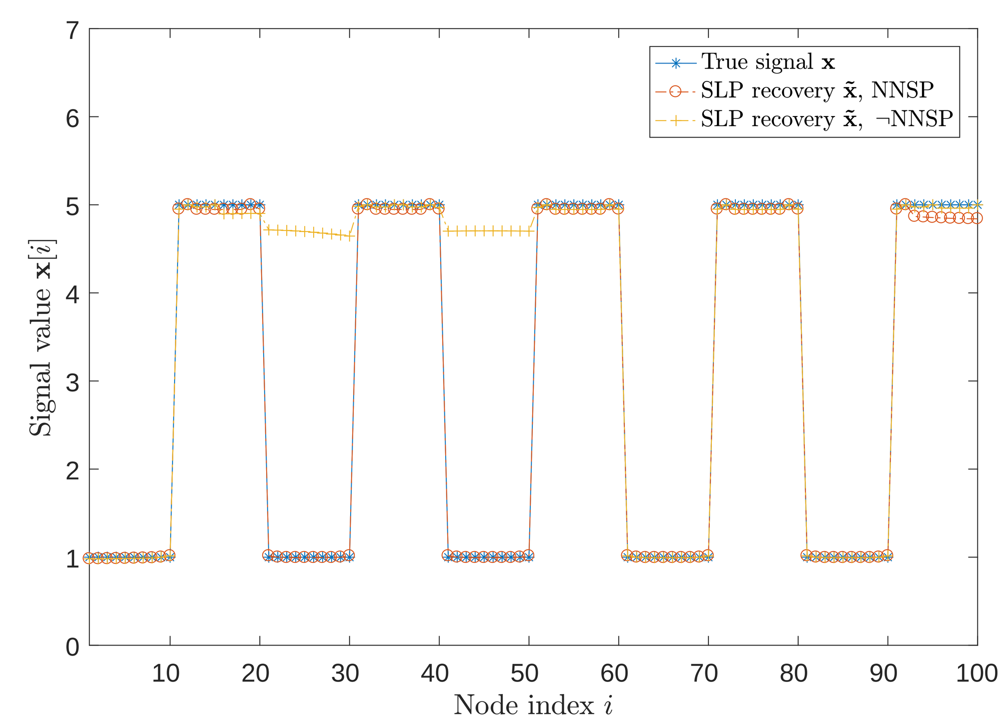
NNSP implies accurate recovery by SLP

Theorem 3 *If the sampling set $\mathcal{M}$ satisfies NNSP, then any solution $\hat{\mathbf{x}}$ of* (1) *satisfies*

$$\|\hat{\mathbf{x}}[\cdot] - \mathbf{x}[\cdot]\|_{TV} \leq 6 \min_{\mathbf{a}_{\mathcal{C}} \in \mathbb{R}^{\mathcal{V}}} \|\mathbf{x}[\cdot] - \sum_{\mathcal{C}=1}^{|\mathcal{F}|} \mathbf{a}_{\mathcal{C}} \mathcal{I}_{\mathcal{C}}[\cdot]\|_{TV}$$

for any clustered graph signal $\mathbf{x} \in \mathbb{R}^{\mathcal{V}}$ of the form (2)

Experiments



If NNSP does not hold, recovery fails

Future research

- Extending our results to networks with certain structure
- Deriving information theoretic limits on required sample size

References

[1] A. Jung and A. Mara and S. Jahromi and A. Hero. "Semi-Supervised Learning via Sparse Label Propagation". JMLR, 2017.

[2] A. Chambolle and T. Pock. "A first-order primal-dual algorithm for convex problems with applications to imaging." J. Math. Imaging Vision, 40(1):120-145, 2011.

[3] J. Kleinberg and E. Tardos. "Algorithm Design". Addison Wesley, 2006.