Non-negative Super-resolution is Stable

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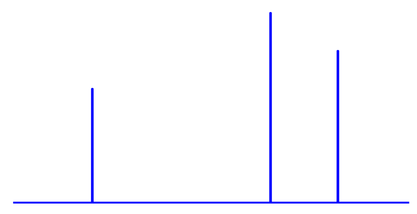
Problem

Consider the problem of localising k non-negative point sources on the interval $[0,1]$, namely finding their locations $t_1, \ldots, t_k$ and magnitudes $a_1, \ldots, a_k \geq 0$ from m noisy samples $y_1, \ldots, y_m$ which consist of the convolution of the input signal with a known kernel ϕ (e.g. Gaussian $\phi(t) = e^{-t^2/\sigma^2}$) and additive noise η bounded by δ. We use the formulation of the problem from [4].

Input signal

x is the discrete measure we want to reconstruct.

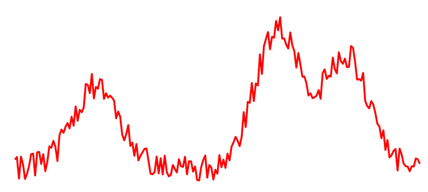


$$x(t) = \sum_{i=1}^{k} a_i \cdot \delta_{t_i},$$

t_i and a_i $(i = 1, \ldots, k)$ are the locations and magnitudes of the point sources.

Measured signal

y_i are the samples we use to reconstruct x.



$$y_j = \int_{[0,1]} \phi(t - s_j) x(\mathrm{d}t) + \eta_j,$$

where s_j $(j = 1, \ldots, m)$ are sampling locations and $\eta = [\eta_1, \ldots, \eta_m]^T$ with $\|\eta\|_2 \leq \delta$ is the noise.

Feasibility problem

We solve the following problem to find x from y

Find $z \geq 0$ subject to

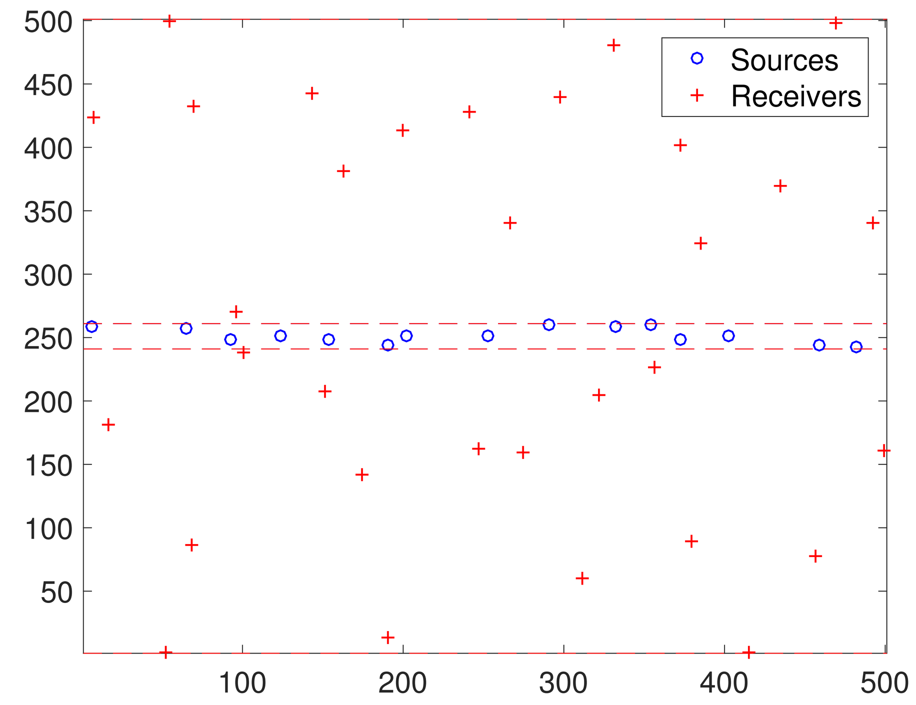
$$\left\| y - \int_{[0,1]} \Phi(t) z(\mathrm{d}t) \right\|_2 \leq \delta \quad (1)$$

where

$$y = [y_1, \ldots, y_m]^T,$$
$$\Phi(t) = [\phi(t - s_1), \ldots, \phi(t - s_m)]^T.$$

Application example

We sample at specific points the sound emitted by ships (sources) traveling along a shipping lane in a 2D region of interest and we want to find their locations and the overall sound level, either in a static or in a dynamic setting. The problem is described in more detail in [2].



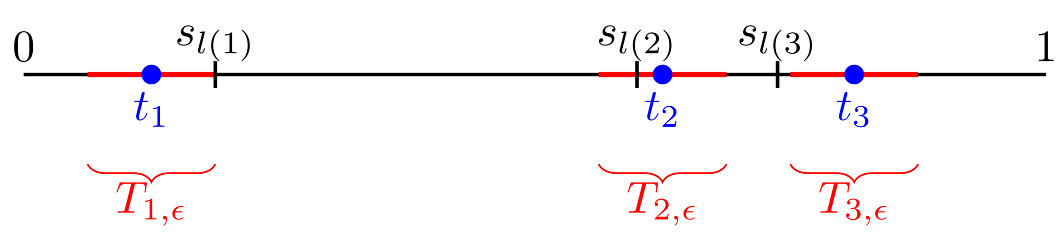
An example of a receiver and source configuration in the region of interest.

Results

Setup

Given the source locations $t_1, \ldots, t_k$, we define:

- $\Delta = \min_{i \neq j} |t_i - t_j|$ is the minimum separation of the sources
- For $\lambda \in [0, \frac{1}{2})$ and for each source t_i, there exists a sampling location $s_{l(i)}$ such that $|s_{l(i)} - t_i| < \lambda\Delta$
- $T_{i,\epsilon} = (t_i - \epsilon, t_i + \epsilon)$ and $T_\epsilon^C = [0,1] \setminus \cup_{i=1}^k T_{i,\epsilon}$ for some $0 < \epsilon < \Delta/2$.



For a solution $\hat{x}$ to the Feasibility Problem (1) and the true measure x, we show that the error in $T_{i,\epsilon}$ is bounded and proportional to the noise level δ and ϵ.

Theorem 1 - General kernel

Let $\hat{x}$ be a solution of (1). If $m \geq 2k+1$, ϕ is Lipschitz continuous and, for Δ, λ and ϵ defined previously, we have

$$\phi(\lambda\Delta) \geq 2\phi(\Delta - \lambda\Delta) + \frac{2}{\Delta} \int_{\Delta-\lambda\Delta}^{1/2-\lambda\Delta} \phi(x)\,\mathrm{d}x$$

then, for all $i \in 1, \ldots, k$,

$$\left| \int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i \right| \leq [2\left(1 + \frac{\phi^\infty \|b\|_2}{\bar{f}}\right) \cdot \delta$$
$$+ L\|\hat{x}\|_{TV} \cdot \epsilon] \|A^{-1}\|_\infty$$

where

- L is the Lipschitz constant of ϕ,
- $\phi^\infty = \max_{t \in [0,1]} |\phi(t)|$,
- b is the vector of coefficients of the dual certificate, and $\bar{f}$ is involved in its construction, see below,
- A is the sampling matrix defined below.

Theorem 2 - Gaussian kernel

Let $\hat{x}$ be a solution of (1) with $\phi(t) = e^{-x^2/\sigma^2}$. If $m \geq 2k+2$ and for every source we have two samples s, s' such that $|s - t_i| \leq \eta$ and $|s - s'| \leq \eta$ with $\eta \leq \sigma^2$. If $\sigma < \frac{1}{\sqrt{3}}, \Delta(T) > \sigma\sqrt{\log\frac{5}{\sigma^2}}$, then:

$$\left| \int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i \right| \leq \left[(c_1 + F_2) \cdot \delta + c_2 \frac{\|\hat{x}\|_{TV}}{\sigma^2} \cdot \epsilon \right] F_3,$$

where $F_2(k, \Delta(T), 1/\sigma, 1/\epsilon) < c_3 \frac{kC_2(1/\epsilon)}{\sigma^2} \left[\frac{c_4}{\sigma^6(1-3\sigma^2)^2} \right]^k$, $F_3(\Delta(T), \sigma, \lambda) < c_5$ and the universal constants c_1, c_2, c_3, c_4, c_5 and $C_2(1/\epsilon)$ are given in [1].

Implications of Theorem 2

- Any solution consistent with measurements within δ is within $\delta + \epsilon$ of the true measure in the appropriate notion.
- As $\delta, \epsilon \to 0$, the measure x is the unique solution to (1).
- Similar results were previously known only for specific algorithms (e.g. TV-norm minimisation).

Proof ideas

Dual certificate

We use the notion of dual certificate to show that the original signal x is the unique solution of problem (1) with $\delta = 0$, then extend this notion to $\delta > 0$ to bound the error between the true and the estimated solution. Using ideas from [3], we show that the dual certificate exists if ϕ with different shifts forms a T-System and we construct it. The dual certificate is a function:

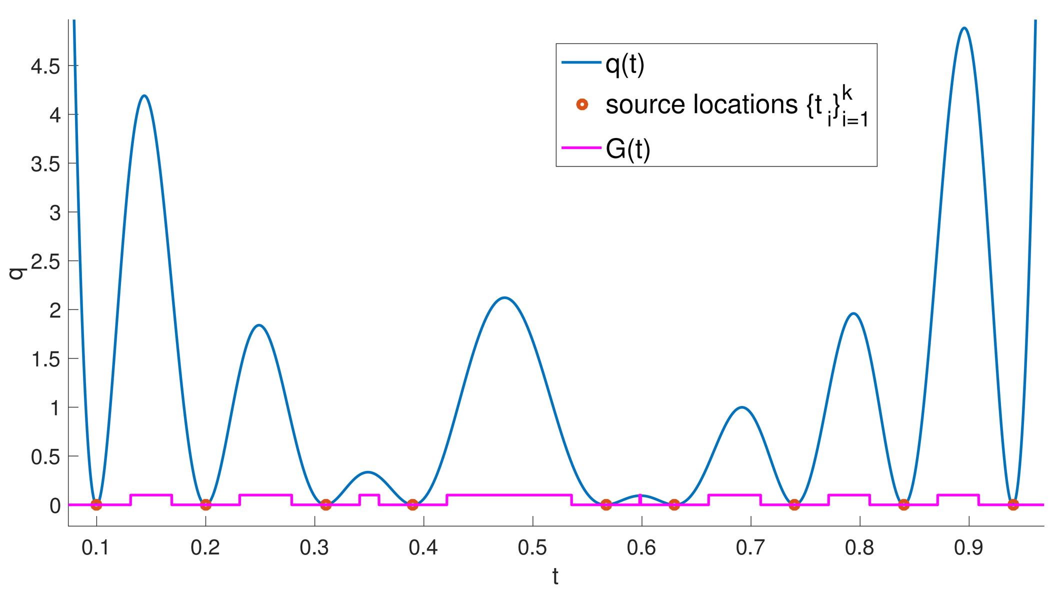
$$q(t) = \sum_{j=1}^{k} b_j \phi(t - s_j)$$

such that

- $q(t_i) = 0, \forall i = 1, \ldots, k$,
- $q(t) > 0, \forall t \neq t_i$.
- $q(t) \geq G(t), \forall t \in [0,1]$

where, for some $\bar{f} > 0$,

$$G(t) = \begin{cases} \bar{f}, & \text{if } t \in T_\epsilon^C, \\ 0, & \text{otherwise.} \end{cases}$$



The shape of the dual certificate depends on the locations of the sources and the samples.

Sampling matrix

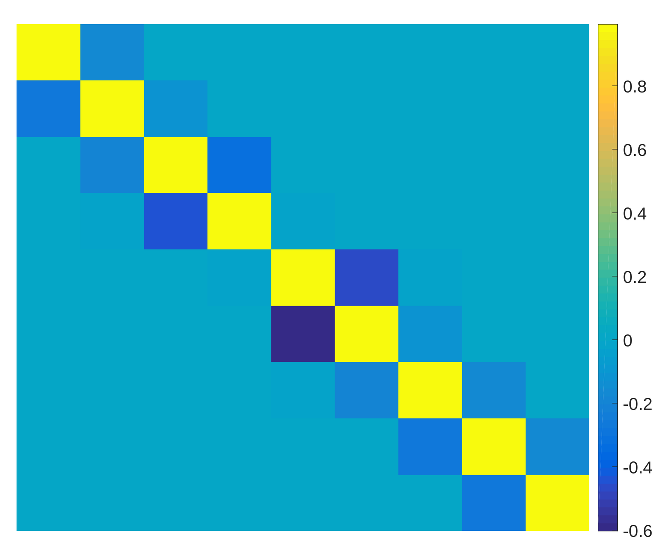
Another important idea is that of diagonally dominance of the matrices A and B defined below. The condition that A is diagonally dominant is related to how close to each source we need the samples to be for a general ϕ in Theorem 1. The determinant of the matrix B appears in the factor F_2 in Theorem 2 for the Gaussian kernel. We bound it by considering bounds on the eigenvalues of B.

$$A = (a_{ij}) \in \mathbb{R}^{k \times k}$$

$$a_{ij} = \begin{cases} |\phi(s_{l(i)} - t_j)|, & \text{if } i = j, \\ -|\phi(s_{l(i)} - t_j)|, & \text{otherwise.} \end{cases}$$

$$B = (B_{ij}) \in \mathbb{R}^{2k \times 2k}$$

$$B_{ij} = \begin{bmatrix} \phi(t_i - t_j) & -\phi'(t_i - t_j) \\ \phi'(t_i - t_j) & -\phi''(t_i - t_j) \end{bmatrix}$$



Example of matrix A.

Example of matrix B.

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Acknowledgments

This project is in partnership with Dr Peter Harris and Dr Stephane Chretien (National Physical Laboratory), who contributed the application problem.

EPSRC Engineering and Physical Sciences Research Council — NPL National Physical Laboratory