# Can Hardware Distortion Correlation be Neglected When Analyzing Uplink SE in Massive MIMO?


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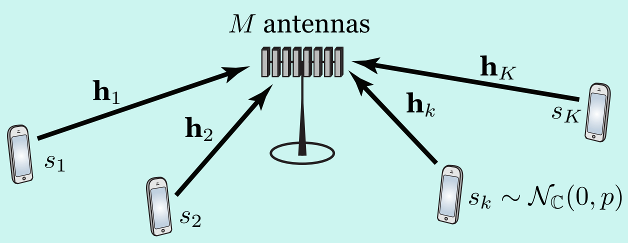
## Background

The received signals at a Massive MIMO base station (BS) are correlated between the antennas

- Hardware distortion also becomes correlated
- But the correlation is often† neglected when analyzing the spectral efficiency (SE)
- Recent works call this approximation "physically inaccurate" — but does it lead to inaccurate results?

**No**, we prove that the approximation errors are small

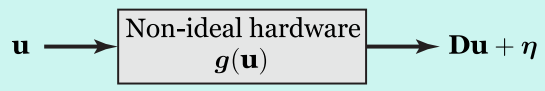
## Uplink System Model




Received signal (at antenna inputs):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix} = [\mathbf{h}_1 \ \ldots \ \mathbf{h}_K] \begin{bmatrix} s_1 \\ \vdots \\ s_K \end{bmatrix} = \mathbf{H}\mathbf{s}$$

Received signal is distorted by the hardware:




- Arbitrary function $\boldsymbol{g}(\mathbf{u}) = [g_1(u_1) \ldots g_M(u_M)]^{\mathrm{T}}$

Conditional statistics for given channel realization $\mathbf{H}$:

- $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_M)$ and $d_m = \frac{\mathbb{E}_{|\mathbf{H}}\{g_m(u_m)u_m^*\}}{\mathbb{E}_{|\mathbf{H}}\{|u_m|^2\}}$
- Distortion $\boldsymbol{\eta} \in \mathbb{C}^M$ is uncorrelated with $\mathbf{u}$ and has non-diagonal correlation matrix $\mathbf{C}_{\eta\eta} = \mathbb{E}_{|\mathbf{H}}\{\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{H}}\}$

> Output of non-ideal hardware is a scaled version of input $\mathbf{u}$ plus distortion $\boldsymbol{\eta}$ that has correlated elements but is uncorrelated with $\mathbf{u}$.

## Spectral Efficiency

Noisy signal used for detection:

$$\mathbf{y} = \boldsymbol{g}(\mathbf{u}) + \mathbf{n} = \mathbf{D}\mathbf{u} + \boldsymbol{\eta} + \mathbf{n}$$

- Noise: $\mathbf{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \sigma^2\mathbf{I}_M)$

Hardware impairments at user-side ($\kappa \in [0,1]$):

- $s_k = \varsigma_k + \omega_k$, with desired signal $\varsigma_k \sim \mathcal{N}_{\mathbb{C}}(0, \kappa p)$ and transmitter distortion $\omega_k \sim \mathcal{N}_{\mathbb{C}}(0, (1-\kappa)p)$

With perfect CSI and treating interference/distortion as noise, the SE for user $k$ is $\mathbb{E}_{\mathbf{H}}\{\log_2(1+\gamma_k')\}$, $\gamma_k' =$

$$\frac{\kappa p |\mathbf{h}_k^{\mathrm{H}}\mathbf{D}^{\mathrm{H}}\mathbf{v}_k|^2}{\sum\limits_{i \neq k} p|\mathbf{h}_i^{\mathrm{H}}\mathbf{D}^{\mathrm{H}}\mathbf{v}_k|^2 + \mathbf{v}_k^{\mathrm{H}}\mathbf{C}_{\eta\eta}\mathbf{v}_k + (1-\kappa)p|\mathbf{h}_k^{\mathrm{H}}\mathbf{D}^{\mathrm{H}}\mathbf{v}_k|^2 + \sigma^2\|\mathbf{v}_k\|^2}$$

- Receive combining vector $\mathbf{v}_k$
- BS distortion: $\mathbf{v}_k^{\mathrm{H}}\mathbf{C}_{\eta\eta}\mathbf{v}_k$
- User distortion: $(1-\kappa)p|\mathbf{h}_k^{\mathrm{H}}\mathbf{D}^{\mathrm{H}}\mathbf{v}_k|^2$

SE maximized by *distortion-aware minimum-mean squared error* (DA-MMSE) combining:

$$\mathbf{v}_k = p\left(\sum_{i=1, i\neq k}^{K} p\mathbf{D}\mathbf{h}_i\mathbf{h}_i^{\mathrm{H}}\mathbf{D}^{\mathrm{H}} + \mathbf{C}_{\eta\eta} + \sigma^2\mathbf{I}_M\right)^{-1}\mathbf{D}\mathbf{h}_k.$$

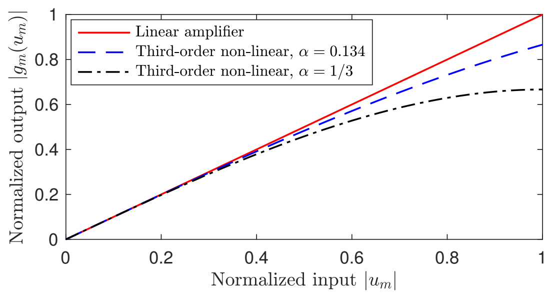
> The optimal receive combining takes the BS distortion correlation into account, but is unaffected by the user distortion.

## Quantifying Impact of Non-Linearities

Model the low-noise amplifier as third-order strictly memoryless non-linear function

$$g_m(u_m) = u_m - a_m|u_m|^2u_m, \quad m = 1, \ldots, M$$

where $a_m = \frac{\alpha}{b_{\mathrm{off}}\mathbb{E}\{|u_m|^2\}}$, $\alpha$ characterizes saturation level, and $b_{\mathrm{off}}$ is the backoff




We obtain $\mathbf{D}$ and $\mathbf{C}_{\eta\eta}$: ($\odot$ = elementwise product)

$$\mathbf{D} = \mathbf{I}_M - 2\mathbf{A} \odot \mathbf{C}_{uu}$$
$$\mathbf{C}_{\eta\eta} = 2\mathbf{A}\left(\mathbf{C}_{uu} \odot \mathbf{C}_{uu}^* \odot \mathbf{C}_{uu}\right)\mathbf{A}$$

with $\mathbf{C}_{uu} = \mathbb{E}_{|\mathbf{H}}\{\mathbf{u}\mathbf{u}^{\mathrm{H}}\} = p\mathbf{H}\mathbf{H}^{\mathrm{H}}$, $\mathbf{A} = \mathrm{diag}(a_1, \ldots, a_M)$

## Distortion Vectors Less Correlated Than Signals

Correlation coefficient for signals $u_i$ and $u_j$:

$$\xi_{u_iu_j} = \frac{\rho_{ij}}{\sqrt{\rho_{ii}\rho_{jj}}} \in [0,1]$$

where $\rho_{ij} = \mathbb{E}_{|\mathbf{H}}\{u_iu_j^*\} = [\mathbf{C}_{uu}]_{ij}$

Correlation coefficient for distortion terms $\eta_i$ and $\eta_j$:

$$\xi_{\eta_i\eta_j} = \frac{\mathbb{E}_{|\mathbf{H}}\{\eta_i\eta_j^*\}}{\sqrt{\mathbb{E}_{|\mathbf{H}}\{|\eta_i|^2\}\mathbb{E}_{|\mathbf{H}}\{|\eta_j|^2\}}} = |\xi_{u_iu_j}|^2\xi_{u_iu_j}$$

> The distortion terms are less correlated than the corresponding signal terms, since
>
> $$|\xi_{\eta_i\eta_j}| = |\xi_{u_iu_j}|^3 \leq |\xi_{u_iu_j}|.$$

## What if the Distortion Correlation is Neglected?

If distortion correlation is low: analytically tractable to neglect it. Use $\mathbf{C}_{\eta\eta}^{\mathrm{diag}} = \mathbf{C}_{\eta\eta} \odot \mathbf{I}_M$ instead of $\mathbf{C}_{\eta\eta}$?

**Assumption:** i.i.d. Rayleigh fading channels $\mathbf{h}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_M)$ for $k = 1, \ldots, K$

> Consider maximum ratio (MR) combining $\mathbf{v}_k = \mathbf{h}_k/\sqrt{\mathbb{E}\{\|\mathbf{h}_k\|^2\}}$, then
>
> $$\mathbb{E}\{\mathbf{v}_k^{\mathrm{H}}\mathbf{C}_{\eta\eta}\mathbf{v}_k\} = c\left(K + 6 + \frac{9}{K} + \frac{4+2M(K+1)}{K^2}\right)$$
> $$\approx \mathbb{E}\{\mathbf{v}_k^{\mathrm{H}}\mathbf{C}_{\eta\eta}^{\mathrm{diag}}\mathbf{v}_k\} = c\left(K + 6 + \frac{11}{K} + \frac{6}{K^2}\right)$$
>
> where $c = \frac{2\alpha^2 p}{b_{\mathrm{off}}^2}$ and the approximation neglects the distortion correlation.

The average distortion power is larger when the BS distortion is correlated:

$$\frac{\mathbb{E}\{\mathbf{v}_k^{\mathrm{H}}\mathbf{C}_{\eta\eta}\mathbf{v}_k\}}{\mathbb{E}\{\mathbf{v}_k^{\mathrm{H}}\mathbf{C}_{\eta\eta}^{\mathrm{diag}}\mathbf{v}_k\}} = 1 + \frac{2(M-1)}{(K+2)(K+3)}$$

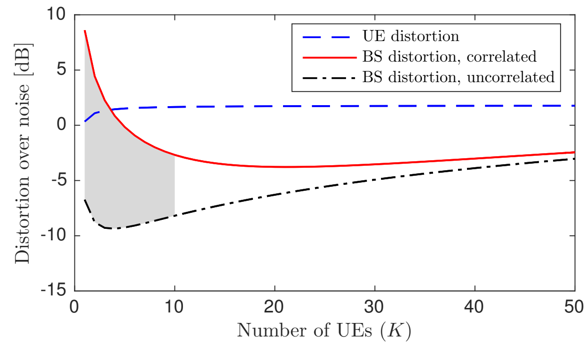
Second term grows as $M$ and decays as $1/K^2$

> Under same assumptions, average user distortion is $(1-\kappa)p\mathbb{E}\{|\mathbf{h}_k^{\mathrm{H}}\mathbf{D}^{\mathrm{H}}\mathbf{v}_k|^2\}$ with
>
> $$\mathbb{E}\{|\mathbf{h}_k^{\mathrm{H}}\mathbf{D}^{\mathrm{H}}\mathbf{v}_k|^2\} = (M+1) - \frac{4\alpha(MK+K+M+3)}{b_{\mathrm{off}}K} + \frac{4\alpha^2(MK^2+8K+11+2MK+K^2+M)}{b_{\mathrm{off}}^2K^2}$$

## Relative Size of BS and User Distortion

**Massive MIMO simulation setup**: $M = 200$, $\alpha = 1/3$, $b_{\mathrm{off}} = 7\,\mathrm{dB}$, $\kappa = 0.99$, and a signal-to-noise ratio (SNR) of $p/\sigma^2 = 0\,\mathrm{dB}$

- BS hardware is "worse" than the user hardware (larger signal-to-distortion power ratio)



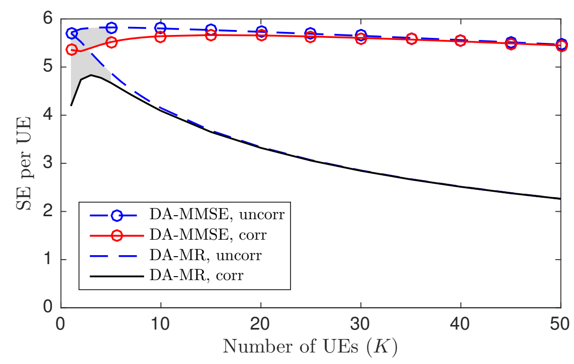

- Distortion correlation has a huge impact on BS distortion when there are few users
- Gap reduces from 15.3 to 5.5 dB in shaded area
- The correlated BS distortion is only the dominant factor for $K \leq 3$, since some terms reduce as $1/K^2$

> The correlation of the BS distortion reduces with $K$. The BS distortion eventually has a smaller impact than the user distortion, which doesn't reduce with $K$.

## SE With and Without Distortion Correlation

We compute SE using $\mathbf{C}_{\eta\eta}$ and $\mathbf{C}_{\eta\eta}^{\mathrm{diag}}$

We compare DA-MMSE and distortion-aware MR (DA-MR) combining ($\mathbf{v}_k = \mathbf{D}\mathbf{h}_k/\|\mathbf{D}\mathbf{h}_k\|$)




- Choice of combining scheme has a large impact
- Approximation error is negligible for $K \geq 5$ with both schemes
- For $K < 5$, the shaded gap only ranges from 6.7% to 5.5% for DA-MMSE

> The distortion correlation has negligible impact on the uplink SE in the studied Massive MIMO scenario; that is, i.i.d. Rayleigh fading and equal SNRs for all users.

## Conclusion

**Yes**, hardware distortion correlation can be neglected when analyzing uplink SE in Massive MIMO!

Our journal paper proves the conclusion with

- Different channel models and varying SNRs
- Quantization distortion
- Imperfect CSI and as $M \to \infty$

In practice, frequency-selective fading and compensation algorithms further supports the conclusion

**But**, one can create setups (ideal user hardware, free-space propagation) when the correlation is influential

†Our book "Massive MIMO Networks: Spectral, Energy, and Hardware Efficiency" reviews how to quantify the SE with uncorrelated distortion. We would happily give you a copy!