On the tradeoff between rate and pairwise error performance of Alamouti and $SP(2)$ space-time block codes

Salime Bameri*, Ramy H. Gohary†, Siamak Talebi*, and Ioannis Lambadaris†

*Electrical Engineering Dep., Shahid Bahonar University of Kerman, Kerman, Iran and †Systems and Computer Engineering Dept., Carleton University, Ottawa, ON, Canada

25–28 June 2018 | IEEE International Workshop on Signal Processing Advances in Wireless Communications (SPAWC) 2018 | Kalamata, Greece

Introduction, Related Work and Objective

- Multiplexing gain: pertains to gradient at which transmission rate supported by system increases with logarithm of SNR.
- Diversity gain: pertains to rate at which PEP decays with logarithm of SNR.
- Both gains are highly desirable but their individual maxima cannot be achieved simultaneously.
- A fundamental tradeoff whereby an increase in one incurs a strict decrease in the other.
- **Related work**:
  - A new asymptotic upper bound on the symbol error rate of STBCs is derived. This bound is based on the distance spectra of the equivalent Euclidean codes introduced (Geyer et al'2015).
  - An analysis characterized the diversity-multiplexing tradeoff for particular STBCs, but does not characterize the tradeoff between their actual PEPs and transmission rates (Dembo et al'1992).
- **Objective**:
  - Derive a tradeoff between actual PEP, rather than diversity gain, and actual transmission rate, rather than multiplexing gain, for two specific STBCs: Alamouti and $\boldsymbol{SP}(2)$ codes.

System Model and System Parameters

- MIMO system with N_t transmit antennas and N_r receive antennas.
- Quasi static fading channels which are i.i.d complex Gaussian distributed with zero mean and unit variance.
- Transmitter sends information in blocks of T consecutive time slots.
- **Received signal**:

$$\boldsymbol{Y} = \sqrt{\frac{P}{N_t}}\boldsymbol{SH} + \boldsymbol{N}.$$

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- Transmitted power: P, transmitted codeword: $\boldsymbol{S} \in \mathbb{C}^{T\times N_t}$, channel matrix: $\boldsymbol{H} \in \mathbb{C}^{N_t\times N_r}$, and additive noise: $\boldsymbol{N} \in \mathbb{C}^{T\times N_r}$.
- **ML detector**:

$$\hat{\boldsymbol{S}} = \arg\min_{\boldsymbol{S}} \|\boldsymbol{Y} - \sqrt{\frac{P}{N_t}}\boldsymbol{SH}\|_F^2.$$

- **PEP**, using ML detector when $\frac{P}{4N_0} \gg 1$:

$$\Pr(\boldsymbol{S}_i \to \boldsymbol{S}_j) \leq \left(\frac{P}{4N_0}\right)^{-N_tN_r} \det(\boldsymbol{\Omega}_{ij})^{-N_r}, \quad \boldsymbol{\Omega}_{ij} = (\boldsymbol{S}_i - \boldsymbol{S}_j)^\dagger(\boldsymbol{S}_i - \boldsymbol{S}_j).$$

- For Alamouti and $\boldsymbol{SP}(2)$ STBCs: $\boldsymbol{\Omega}_{ij}$ is full rank.

PEP and Rate Tradeoff for Alamouti Code

- Derive bounds on right side hand of PEP, PEP and transmission rate R tradeoff:
- **Alamouti STBC**:

$$\boldsymbol{S}_r = \frac{1}{\sqrt{2}}\begin{bmatrix} s_1^{(r)} & s_2^{(r)} \\ -s_2^{(r)} & s_1^{(r)} \end{bmatrix}.$$

  - **Theorem 1**: PEP upper bound when symbols are chosen from uniform M-PSK constellation:

$$\text{PEP}_1 = \Pr(\boldsymbol{S}_i - \boldsymbol{S}_j) \leq \left(\frac{P}{2N_0}\right)^{-2N_r} \frac{1}{\sin^{4N_r}(\pi 2^{-R})}, \quad R = \log_2 M.$$

PEP and Rate Tradeoff for $SP(2)$ Code

- $\boldsymbol{SP}(2)$ **STBC**:

$$\boldsymbol{S}_r = \frac{1}{2}\begin{bmatrix} \boldsymbol{U}_r\boldsymbol{V}_r & \boldsymbol{U}_r\overline{\boldsymbol{V}}_r \\ -\overline{\boldsymbol{U}}_r\boldsymbol{V}_r & \overline{\boldsymbol{U}}_r\overline{\boldsymbol{V}}_r \end{bmatrix}, \quad \boldsymbol{U}_r = \begin{bmatrix} e^{j\frac{2\pi k_r}{M_1}} & e^{j\frac{2\pi l_r}{M_1}} \\ -e^{-j\frac{2\pi l_r}{M_1}} & e^{-j\frac{2\pi k_r}{M_1}} \end{bmatrix}, \quad \boldsymbol{V}_r = \begin{bmatrix} e^{j\frac{2\pi m_r}{M_2}} & e^{j\frac{2\pi n_r}{M_2}} \\ -e^{-j\frac{2\pi n_r}{M_2}} & e^{-j\frac{2\pi m_r}{M_2}} \end{bmatrix}.$$

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  - Integers $k_r, l_r \in \{0, \dots, M_1 - 1\}, m_r, n_r \in \{0, \dots, M_2 - 1\}$, M_1 and M_2 denote the cardinality of PSK constellations underlying the $\boldsymbol{SP}(2)$ code.
  - Choosing M_1 and M_2 to be odd and co-prime ensures that the $\boldsymbol{SP}(2)$ STBC achieves maximal diversity.
  - **Theorem 2**: PEP upper bound when M_1 and M_2 are co-prime:

$$\text{PEP}_2 = \Pr(\boldsymbol{S}_i - \boldsymbol{S}_j) \leq \left(\frac{P}{N_0}\right)^{-4N_r} \frac{1}{\sin^{4N_r}(\pi 2^{-2R})}, \quad R = \frac{1}{2}\log_2(M_1M_2)$$

Proof of Theorem 2

- Proof of Theorem 2 involves:
  - **Lemma 1**: Two constellations, M_1-PSK and M_2-PSK, where M_1 and M_2 are two co-prime integers. The minimum non-zero angle between any two symbols drawn from these constellations is $|\Delta\theta_{\min}| = \frac{2\pi}{M_1M_2}$.
    - Fermat' s Little Theorem in Number Theory is used to prove this lemma (Adams et al'1976).
    - Fermat' s Little Theorem: if p is a prime number, then for any integer a:

$$a^p \equiv a \pmod{p}.$$

  - **Lemma 2**: Let M_1 and M_2 be co-prime, then for any integer $k \in \{1, \dots, M_1 - 1\}$, $|\sin(\frac{4\pi k}{M_1})| > |\sin(\frac{2\pi}{M_1M_2})|$ and $|\sin(\frac{2\pi k}{M_1})| > |\sin(\frac{2\pi}{M_1M_2})|$.

Comments on The Results

- The expressions for the upper bound on the PEP of Alamouti STBC and $\boldsymbol{SP}(2)$ STBCs expose inherent rate-performance tradeoff exhibited by these codes.
- For a given number of receive antennas, N_r, and a transmit power, P, PEP bound of the $\boldsymbol{SP}(2)$ STBC grows faster with transmission rate R than its Alamouti counterpart.
- This suggests: for fixed N_r and P,
  - at lower rates: it is more beneficial to use $\boldsymbol{SP}(2)$ STBC,
  - at higher rates: Alamouti STBC is more beneficial even though it uses half the number of transmit antennas.

Numerical Result

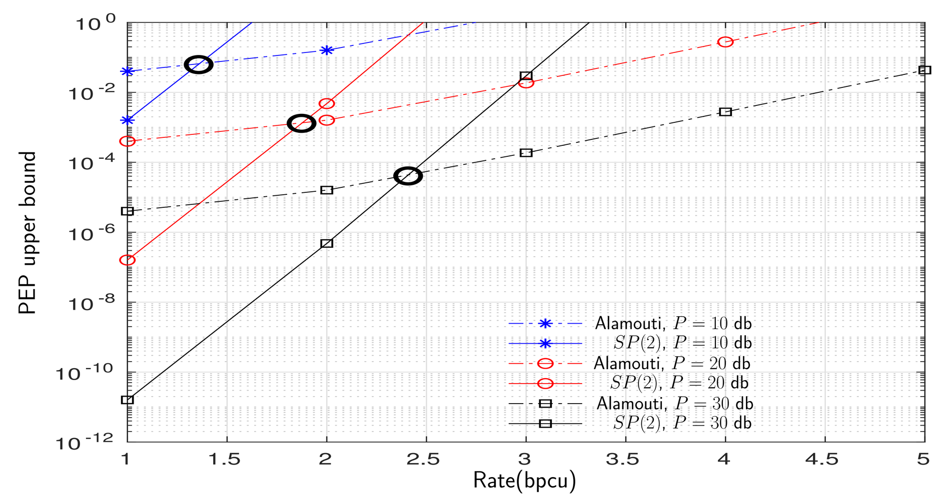

Figure: PEP upper bound vs. rate

Simulation

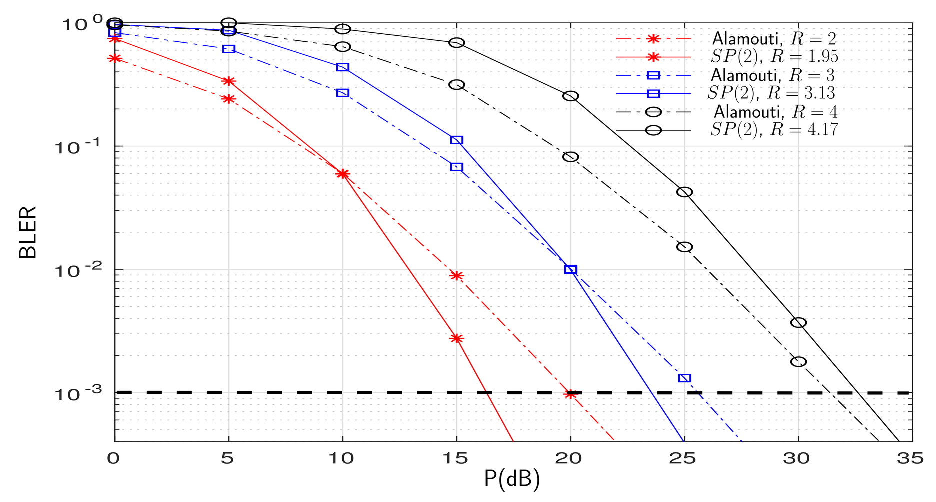

Figure: BLER vs. power.

Conclusion

- Derived upper bounds on PEP of two popular STBCs: Alamouti and $\boldsymbol{SP}(2)$ STBCs.
- These bounds used to obtain a trade-off between transmission rate and PEP achieved by these STBCs.
- Showed that, at high rates, Alamouti STBC outperforms $\boldsymbol{SP}(2)$ STBC, even though $\boldsymbol{SP}(2)$ STBC has twice diversity gain as its Alamouti counterpart.
- Analysis proposed herein might be used to optimize design of the $\boldsymbol{SP}(2)$ code, for example by using an appropriate rotation of constellation.