Impact of Space-Time Covariance Estimation Errors on a Parahermitian Matrix EVD

Connor Delaosa*, Fraser K. Coutts*, Jennifer Pestana†, Stephan Weiss*

* Department of Electronic & Electrical Engineering, University of Strathclyde, Glasgow, Scotland
† Department of Mathematics & Statistics, University of Strathclyde, Glasgow, Scotland

University of Strathclyde Engineering

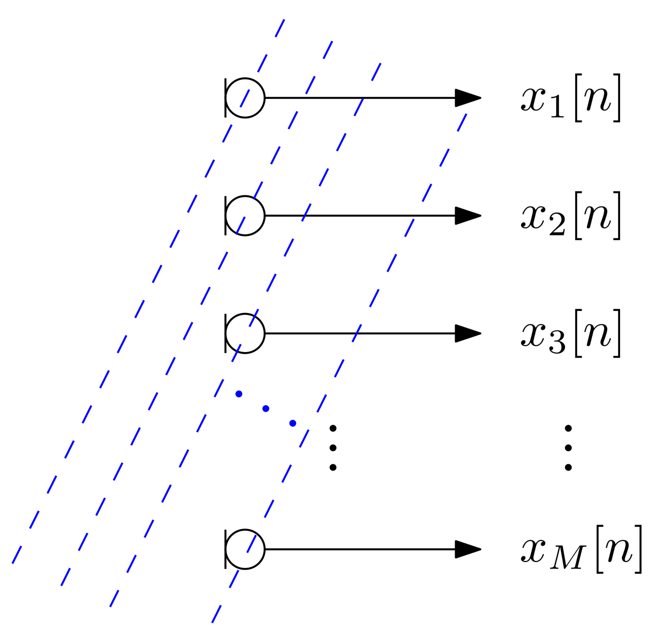
Introduction

In an extension of narrowband array processing to the broadband case, we can utilise the equivalent of the eigen- or singular value decomposition using polynomial matrices. These matrices are key to a number of applications ranging from broadband MIMO systems [1] to angle of arrival estimation [2] and more.

This research aims to investigate the impact of estimation errors in the sample space-time covariance matrix and how this perturbation affects the parahermitian/polynomial matrix eigenvalue decomposition (PEVD).

Space-Time Covariance Matrix and Parahermitian EVD

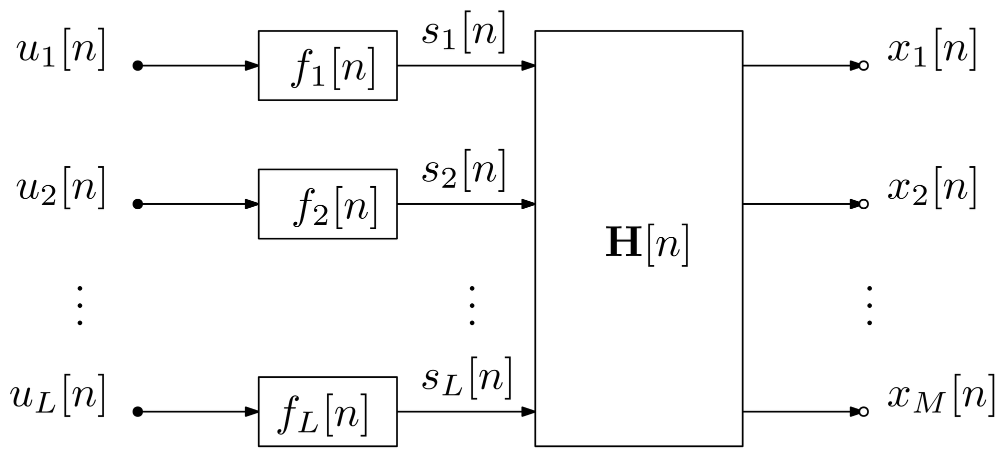
- We assume that an M-element array records data into a vector $\mathbf{x}[n] \in \mathbb{C}^M$



- Ideal space-time covariance matrix: $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$, $\mathbf{R}[\tau] \in \mathbb{C}^{M\times M}$
- $\mathbf{R}[\tau]$ is a matrix of auto- & cross- correlation sequences
- Symmetry: $\mathbf{R}[\tau] = \mathbf{R}^{\mathrm{H}}[-\tau]$
- Cross-spectral density $\boldsymbol{R}(z) = \sum_\tau \mathbf{R}[\tau]z^{-\tau}$ is a polynomial matrix.
- PEVD: $\boldsymbol{R}(z) = \boldsymbol{H}(z)\boldsymbol{\Lambda}(z)\boldsymbol{H}^{\mathrm{P}}(z)$
- Parahermitian: $\boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^*) = \boldsymbol{R}(z)$
- Paraunitary matrix: $\boldsymbol{H}(z)\boldsymbol{H}^{\mathrm{P}}(z) = \mathbf{I}$

Source Model

- To form a valid comparison, a ground truth must be acquired. To describe this ground truth we use the source model:



- The source model assumes that $\mathbf{x}[n]$ is generated by convolutively mixing L independent source signals.
- Source signals $s_\ell[n]$ can be tied to their individual power spectral densities (PSD) via the innovation filters s.t. $S_\ell(z) = F_\ell(z)F_\ell^{\mathrm{P}}(z)$ using the fact that $u_\ell[n]$ are zero mean uncorrelated unit variance Gaussian signals.
- The convolutive mixing matrix $\mathbf{H}[n]$ is described by a network of transfer functions $\boldsymbol{H}(z) : \mathbb{C} \to \mathbb{C}^{M\times L} \bullet\!\!-\!\!\circ\, \mathbf{H}[n]$.
- The cross-spectral density (CSD) matrix is given by $\boldsymbol{R}(z) = \boldsymbol{H}(z)\boldsymbol{S}(z)\boldsymbol{H}^{\mathrm{P}}(z)$ where $\boldsymbol{S}(z) = \mathrm{diag}\{S_1(z), \ldots, S_L(z)\}$.

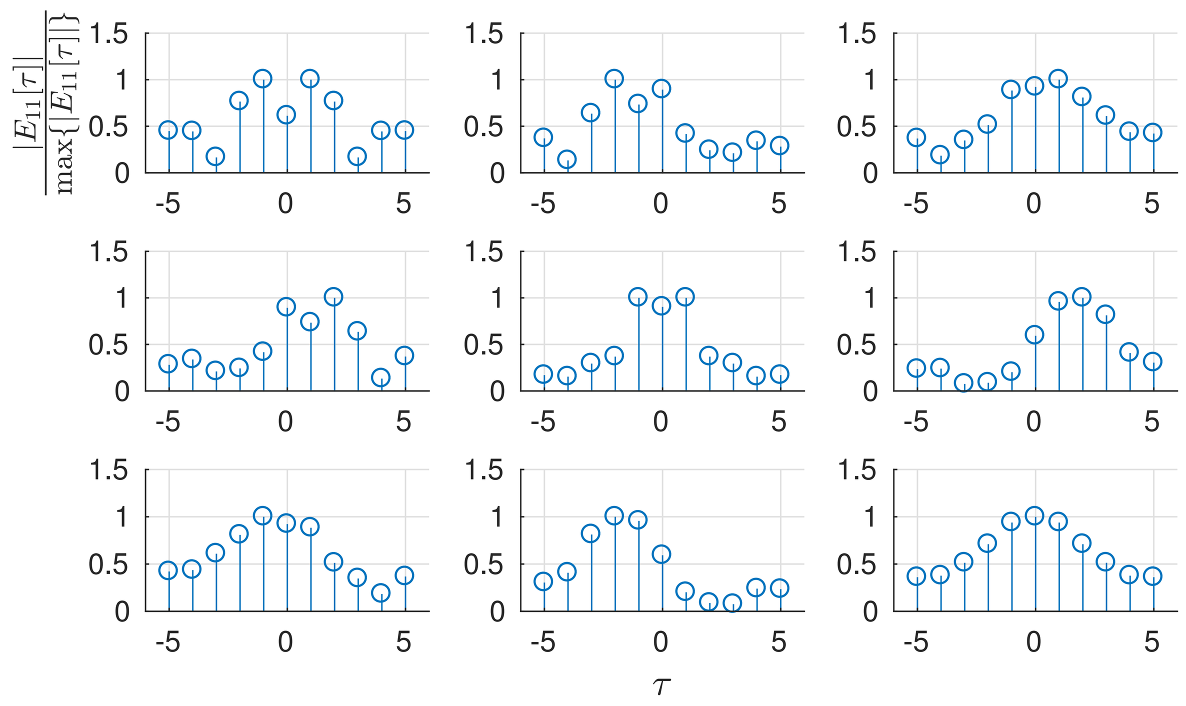
Sample Space-Time Covariance Matrix

In practice we do not have access to unlimited data samples and therefore must estimate $\mathbf{R}[\tau]$ using N samples which results in a sample space-time covariance matrix:

$$\hat{\mathbf{R}}[\tau] = \frac{1}{N-\tau}\sum_{n=\tau}^{N-1}\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]; \qquad (1)$$

- This approach assumes ergodicity which implies that $\lim_{N\to\infty}\hat{\boldsymbol{R}}(z) = \boldsymbol{R}(z)$.
- From this we can define the error between the ground truth and estimation: $\mathbf{E}[\tau] = \hat{\mathbf{R}}[\tau] - \mathbf{R}[\tau]$.
- From the ergodicity and unbiased estimate of (1) we can say that $\lim_{N\to\infty}\mathbf{E}[\tau] = 0,\ \forall\tau$.
- Estimated PEVD: $\hat{\boldsymbol{R}}(z) = \hat{\boldsymbol{H}}(z)\hat{\boldsymbol{\Lambda}}(z)\hat{\boldsymbol{H}}^{\mathrm{P}}(z)$.
- We assume that $N > M$ samples are available so that there is a chance where $\mathrm{rank}\{\hat{\mathbf{R}}[\tau]\} = \mathrm{rank}\{\mathbf{R}[\tau]\}$.

For $N = 100$ and $M = 3$ we provide a normalised version of the matrix $\mathbf{E}[\tau]$:



Perturbation of Eigenvalues and Eigenvectors

When we calculate $\hat{\boldsymbol{R}}(z)$ the corresponding eigenvalues and -vectors are affected as a result. We can evaluate $\hat{\boldsymbol{R}}(z)|_{z=e^{j\Omega}} = \hat{\boldsymbol{R}}(e^{j\Omega})$ and the equivalent in the ground truth case.

- From [3] we gain bounds for the perturbation of the mth eigenvalue if the eigenvalues are ordered from highest to lowest power:

$$\lambda_M\{\boldsymbol{E}(e^{j\Omega_0})\} \le \hat{\lambda}_m(e^{j\Omega_0}) - \lambda_m(e^{j\Omega_0}) \le \lambda_1\{\boldsymbol{E}(e^{j\Omega_0})\}. \qquad (2)$$

- From [4] we relate the difference between the eigenvalues of $\boldsymbol{R}(e^{j\Omega_0})$ & $\hat{\boldsymbol{R}}(e^{j\Omega_0})$ and the spectral-norm of the error $\boldsymbol{E}(e^{j\Omega_0})$:

$$|\hat{\lambda}_m(e^{j\Omega_0}) - \lambda_m(e^{j\Omega_0})| \le \kappa\{\boldsymbol{U}(e^{j\Omega_0})\}\|\boldsymbol{E}(e^{j\Omega_0})\|_2, \qquad (3)$$

where $\kappa\{\boldsymbol{U}(e^{j\Omega_0})\} = 1$ due to the paraunitary property of $\boldsymbol{U}(z)$.

For the eigenvectors, we can characterise the eigenvector subspace perturbation as dependent on the distance between their associated eigenvalues. From this, we can say that the closer that the eigenvalues become to each other the larger the perturbation of the subspace can become.

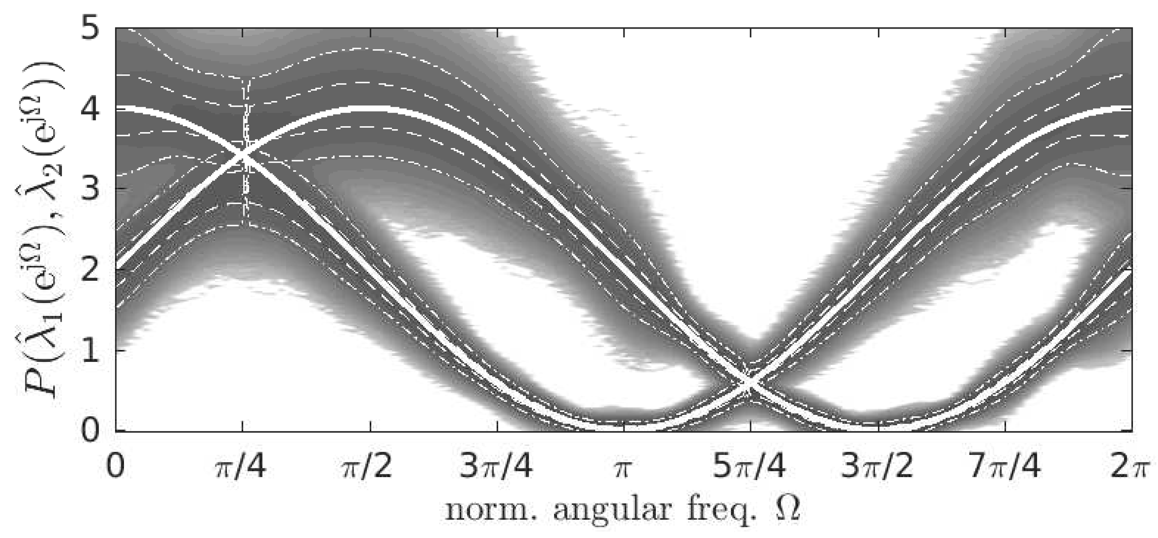
- To quantify these eigenvector perturbations, we define a metric based upon a modified version of the subspace correlation:

$$\gamma_m(e^{j\Omega}) = 1 - |\mathbf{u}_m(e^{j\Omega})\hat{\mathbf{u}}_m^{\mathrm{H}}(e^{j\Omega})|, \quad 0 \le \gamma_m(e^{j\Omega}) \le 1. \qquad (4)$$

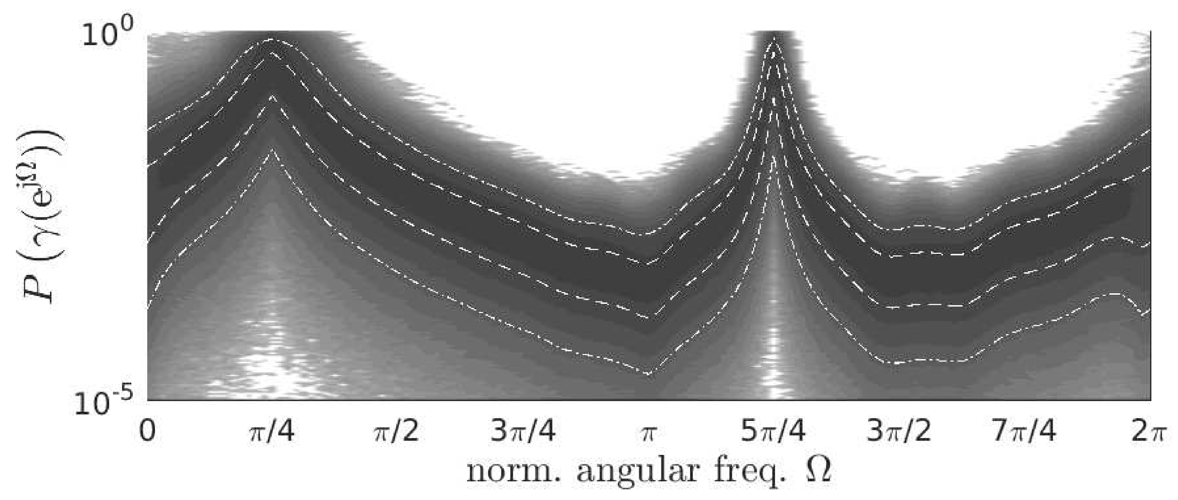
Results

For these results, two models are used: Model 1 uses known eigenvalues that cross at specific frequencies whilst Model 2 is designed to result in well-separated eigenvalues.

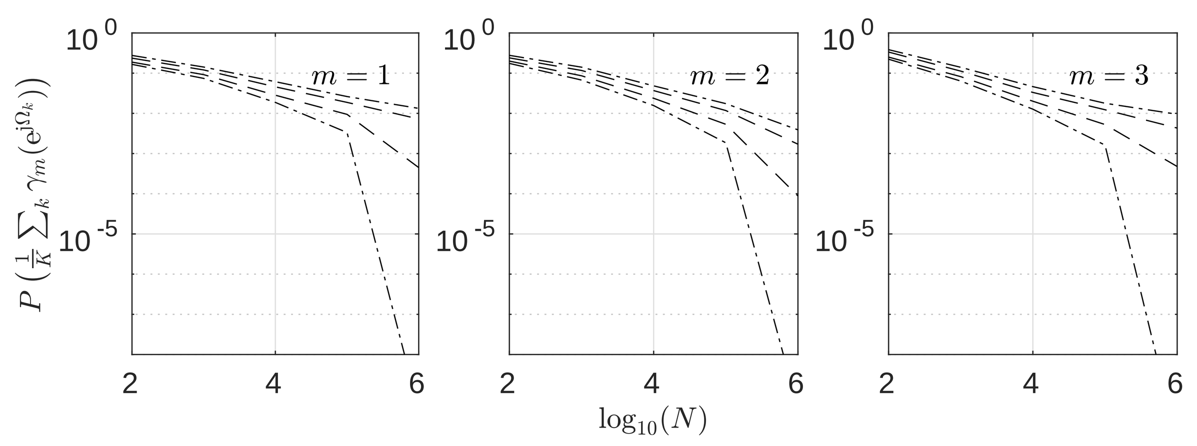
Distribution of $\hat{\boldsymbol{R}}(e^{j\Omega})$ with the eigenvalues of $\boldsymbol{R}(z)$ as a solid line (Model 1) over an ensemble of 10^5 runs, 128 frequency bins for the 5, 25, 75 and 95th percentiles (dotted line) and median (solid line):



Distribution of $\gamma_m(e^{j\Omega})$ (Model 1) with parameters defined above:



Measured distribution of $\gamma_m(e^{j\Omega})$ (Model 2) with percentiles defined above:



Conclusion

When estimating a cross-spectral density matrix from a finite data set, the arising estimation error results in perturbations of the ground truth eigenvalues and eigenvectors of its parahermitian matrix eigenvalue decomposition.

Evaluated on the unit circle, bounds for the perturbation of both quantities can be stated: while the eigenvalue perturbation depends on the estimation error, the eigenvector perturbation additionally depends on the distance between the ground truth eigenvalues. These findings have been demonstrated in and underpinned by a number of simulations.

References

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[2] S. Weiss, M. Alrmah, S. Lambotharan, J. McWhirter, and M. Kaveh. Broadband angle of arrival estimation methods in a polynomial matrix decomposition framework. In *IEEE 5th International Workshop on Computational Advances in Multi–Sensor Adaptive Processing*, pages 109–112, Dec 2013.

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connor.delaosa@strath.ac.uk