Aalto University
School of Electrical Engineering

Symmetric Sparse Linear Array for Active Imaging

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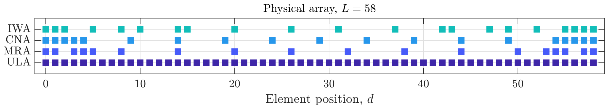
Outline

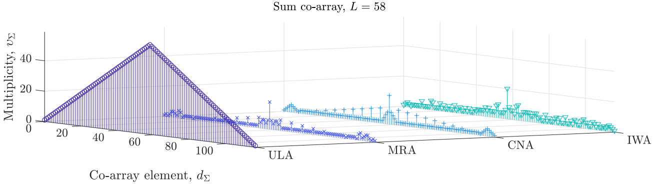
Introduction

- **Motivation:** Increasing demand for ever larger arrays (3D imaging, multi-target tracking, Massive MIMO etc.)
- **Problem:** expensive front-ends/sensors, limited acquisition/processing capability, mutual coupling...
- **Goal:** reduce number of sensor without incurring significant performance loss
- **Solution:** *sparse arrays* utilizing the *co-array*
- **Contribution:** Interleaved Wichmann Array
  - Novel sparse symmetric linear array configuration
  - Contiguous *sum co-array* (proof)
  - Achieves same PSF as ULA of equal aperture
- **Applications:** ultrasound or microwave imaging, radar, indoor localization...

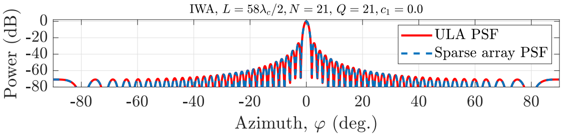
Sum co-array determines achievable PSF



(a) Active linear array configurations with co-located transceivers.



(b) Sum co-arrays



(c) Point spread functions

Signal model for near field active imaging

- Goal: estimate scene reflectivity $\gamma(r, \varphi)$
- Simplifying assumptions:
  - *N* omnidirectional transceivers
  - *K* point targets
  - Frequency independent reflectivity
  - No multipath, clutter or noise

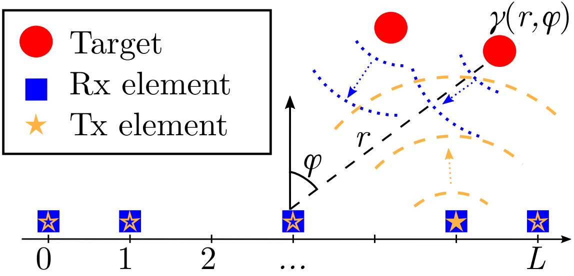


Figure: Active imaging in the plane using a sparse linear array.

Signal model for near field active imaging

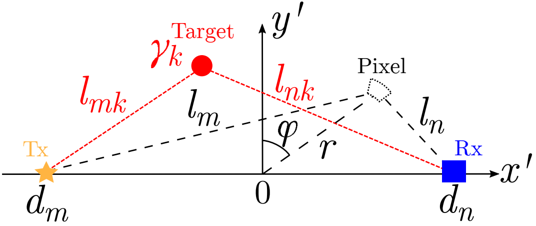


Figure: Element positions are given by d_i, distances by l_i and target reflectivities by γ_k.

- Time diff. between focus delay and prop. delay to target k[1]:

$$\Delta\tau_{mnk} = (l_m + l_n - (l_{mk} + l_{nk}))/c$$

- Reflectivity estimate after beamf. and matched filtering:

$$\hat{\gamma}(r,\varphi) = \sum_{n=1}^{N}\sum_{m=1}^{N} w_{\mathrm{r},n} w_{\mathrm{t},m} \sum_{k=1}^{K} \gamma_k e^{j\omega_c \Delta\tau_{mnk}} R_{ss}(\Delta\tau_{mnk})$$

[1] Dist. from i^{th} element to pixel: $l_i = \sqrt{r^2 + d_i^2 - 2rd_i \sin\varphi}$

Sum co-array

- In the far field ($r, r_k \to \infty$) the delay simplifies to:

$$\Delta\tau_{mnk} = (d_m + d_n)(\sin\varphi_k - \sin\varphi)/c$$

- *Sum co-array* = virtual array determining achievable PSF[2]
- Support: $\mathcal{C}_\Sigma = \{d_\Sigma \mid d_{\Sigma,i} = d_m + d_n\}$
- Multiplicity: $\upsilon_\Sigma(d_{\Sigma,i}) = \sum_{m,n} \mathbb{1}(d_m + d_n = d_{\Sigma,i})$

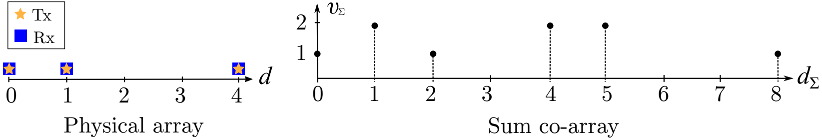


Figure: Example of a sparse array and its sum co-array.

[2] [Hoctor and Kassam, 1990]

Minimum-Redundancy and Wichmann Array

- Minimum-Redundancy Array[3] (MRA)
  - ✓ Optimal (minimizes N, s.t. a contiguous co-array)
  - ✗ Impractical to find for large arrays (search space $\propto 2^L$)
- Wichmann Array[4] (WA)
  - ✓ Optimal WAs are MRAs (empirical observation, not proof)
  - ✓ Closed-form sensor positions
  - ✗ Non-contiguous sum co-array[5]

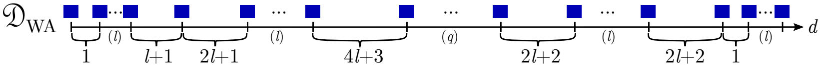


Figure: Wichmann Array. Parameters $l, q \in \mathbb{N}$ control the distance between consecutive elements (braces), and the number of times these are repeated (parenthesis).

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[3] [Moffet, 1968, Hoctor and Kassam, 1996]

[4] [Wichmann, 1963, Pearson et al., 1990, Linebarger et al., 1993]

[5] Counting arg.: $N(N+1)/2 \geq 2L+1$, or asymptotically: $\lim_{L\to\infty} N^2/L \geq 2$. However $\lim_{L\to\infty} N^2/L = 3 < 2$.

Interleaved Wichmann Array (IWA)

Definition (Interleaved Wichmann Array)

Element positions of the IWA are given by $\mathcal{D}_{\text{IWA}} = \mathcal{D}_{\text{WA}} \cup \mathcal{D}_{\text{WA}^-}$.

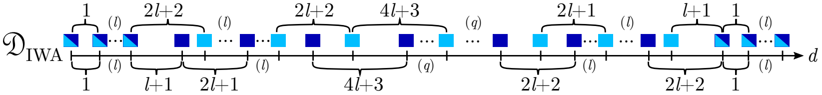


Figure: The IWA is the union of a WA (dark elements) and its mirror image (light elements).

- IWA $=$ superposition of a WA with its mirror image
- Contiguous sum (and difference) co-array guaranteed
  - Follows from symmetry of IWA and diff. co-array of WA
  - Proof up next. . .

Proof of contiguous co-array

Lemma (Co-array of symmetric array)

If $\mathcal{D}$ is mirror symmetric, then $\mathcal{D} + \mathcal{D} = \mathcal{D} - \mathcal{D} +$ const.

Proof.

This follows from the equivalence of the convolution and autocorrelation of a real symmetric function, i.e.
$f(t) = f(-t), f \in \mathbb{R} \Rightarrow f(t) \star f(t) = f(t) * f^*(-t) = f(t) * f(t)$. □

Theorem (Co-array of IWA)

Both $\mathcal{D}_{\text{IWA}} - \mathcal{D}_{\text{IWA}}$ and $\mathcal{D}_{\text{IWA}} + \mathcal{D}_{\text{IWA}}$ are contiguous.

Proof.

This follows directly from the above Lemma, since the WA's difference co-array is contiguous and the IWA is symmetric. □

Optimal IWA parameters

- Parameters $l, q \in \mathbb{N}$ control element positions
- Maximize aperture L, given the no. of elements N:

$$\underset{l,q\in\mathbb{N}}{\text{maximize}} \quad 4l(l+q+2)+3(q+1)$$
$$\text{subject to} \quad N = 2(q+2+3l)$$

☹ Non-convex integer program, however…

☺ …relaxation $l, q \in \mathbb{R}_+$ yields concave objective…

☺ …admitting closed-form solution to original problem[6]:

$$l^\star = \lfloor (2N-9)/16 \rceil$$
$$q^\star = N/2 - 3l^\star - 2$$

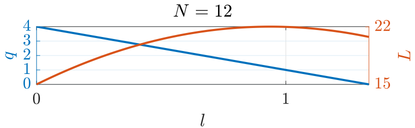
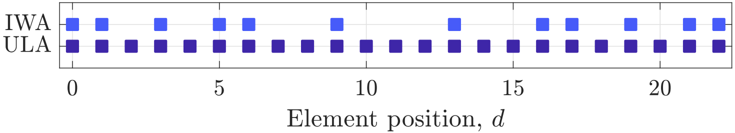


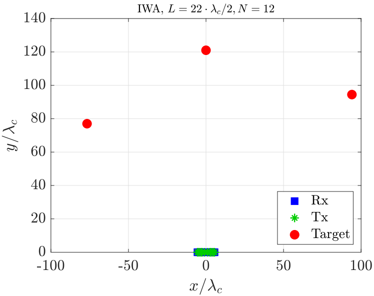
Figure: When $N=12$, $l^\star = q^\star = 1$.

[6] Feasibility: $N=4+2m,\ m\in\mathbb{N} \Rightarrow l^\star, q^\star \in \mathbb{N}$. Optimality: $L=-8l^2+(2N-9)l+3N/2-3 \Rightarrow L(l^\star) \geq L(l\in\mathbb{N})$.

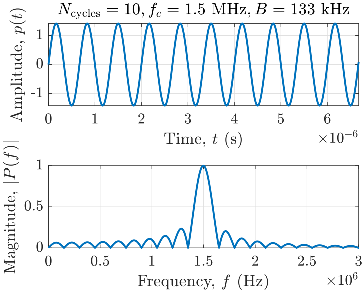
Imaging example



(a) Uniform Linear Array and Interleaved Wichmann Array



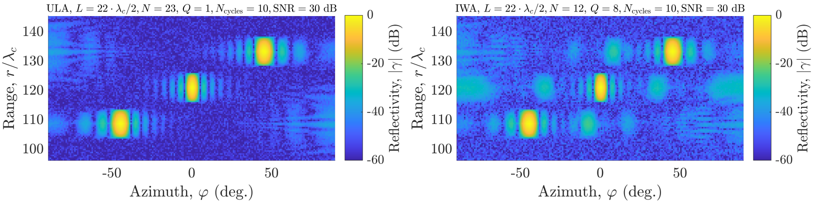
(b) Target scene



(c) Transmitted waveform

Figure: Imaging three targets at a distance of 10-12 array apertures (9% rel. bandwidth)

Comparison with ULA



(a) Uniform Linear Array

(b) Interleaved Wichmann Array

Figure: Images using ULA and IWA (image addition, triangular co-array weighting).

- ✓ Good match close to target
- ✓ 50% fewer elements
- ✗ Grating lobes due to *spatially varying co-array*[7]
- ✗ $-30 \log(12/23) \approx 8.5$ dB lower SNR

[7] [He and Kassam, 2015]

Conclusions

- Introduced the Interleaved Wichmann Array (IWA)
  - Sparse array configuration with co-located transceivers
  - Suitable for both active and passive sensing
- Derived optimal sensor placements of IWA
- Proved that sum and diff. co-array of IWA are contiguous
  - Match PSF of, or resolve same $\#$ of targets as ULA
- Proposed approach can be used to create other symmetric configurations with a contiguous sum co-array

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Backup slides

Signal model for near field active imaging

- Signal received at n^{th} receiver (Tx from m^{th} transmitter):

$$x_{mn}(t) = \sum_{k=1}^{K} \gamma_k s(t - \tau_{mnk}) e^{j\omega_c (t-\tau_{mnk})}$$

- After beamforming:

$$y(t, r, \varphi) = \sum_{n=1}^{N} \sum_{m=1}^{N} w_{r,n} w_{t,m} x_{mn}(t + \tau_{mn})$$

- After matched filtering (estimate of reflectivity):

$$\hat{\gamma}(r, \varphi) = \sum_{n=1}^{N} \sum_{m=1}^{N} w_{r,n} w_{t,m} \sum_{k=1}^{K} \gamma_k e^{j\omega_c \Delta\tau_{mnk}} R_{ss}(\Delta\tau_{mnk})$$

Optimal aperture and number of unit spacings

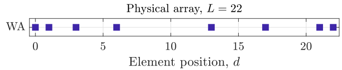
- By definition: $N \geq 4$ and even, $m \in \mathbb{N}$
- Optimal aperture, L:

$$L = (N^2 + 3N - \beta)/8, \text{ where } \beta = \begin{cases} 4, & \text{when } N = 4 + 8m \\ 6, & \text{when } N = 6 + 8m \\ 16, & \text{when } N = 8 + 8m \\ 10, & \text{when } N = 10 + 8m \end{cases}$$

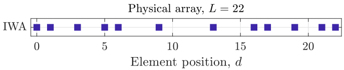
- Optimal number of unit spacings, $\upsilon_\Delta(1)$:

$$\upsilon_\Delta(1) = N/4 + \zeta, \text{ where } \zeta = \begin{cases} 1, & \text{when } N = 4 + 8m \\ 1/2, & \text{when } N = 6 + 8m \\ 0, & \text{when } N = 8 + 8m \\ 3/2, & \text{when } N = 10 + 8m \end{cases}$$

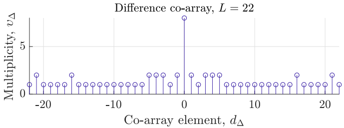
Comparison with WA



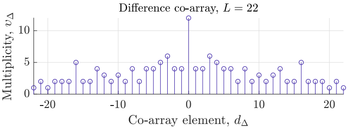
(a) Wichmann Array



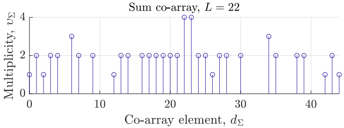
(b) Interleaved Wichmann Array



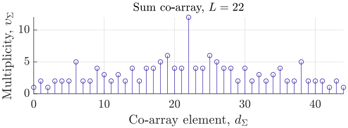
(c) Difference co-array of WA



(d) Difference co-array of IWA



(e) Sum co-array of WA

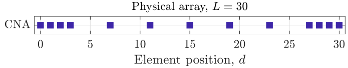


(f) Sum co-array of IWA

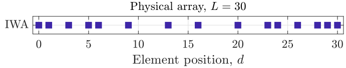
- ✓ Contiguous sum and difference co-array
- ✗ 50% more elements ($\lim_{L\to\infty} N_{\text{IWA}}/N_{\text{WA}} \to \sqrt{8/3} \approx 1.6$) [8]

[8] Counting argument yields $\lim_{L\to\infty} N_+/N_- \to \sqrt{2} \approx 1.4$, since $a+b=b+a$, but $a-b \neq b-a$

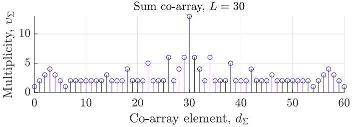
Comparison with CNA[9]



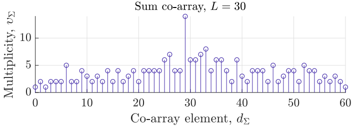
(a) Concatenated Nested Array



(b) Interleaved Wichmann Array



(c) Sum co-array of WA

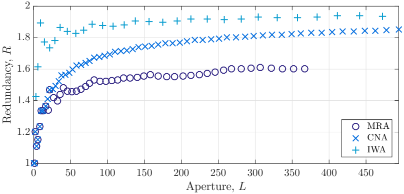


(d) Sum co-array of IWA

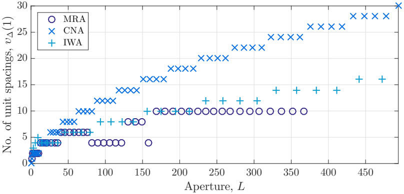
- ✓ one less unit spacing ($\lim_{L\to\infty} v_{\Delta,\text{IWA}}(1)/v_{\Delta,\text{CNA}}(1) \to 0.5$)
- ✗ two more elements (but ✓ $\lim_{L\to\infty} N_{\text{IWA}}/N_{\text{CNA}} \to 1$)

[9] [Rajamäki and Koivunen, 2017]

Comparison of IWA, CNA and MRA



(a) Redundancy



(b) Sparseness

Figure: The IWA has marginally more elements than the CNA for finite apertures, but approximately half the no. of unit spacings.

Image addition

- Physical array has significantly fewer weights than co-array
- Single Tx-Rx weight pair not enough to achieve target PSF
- However, several weightings can match any target function:

$$\mathbf{w}_{\Sigma} = \sum_{q=1}^{Q} \mathbf{w}_{\mathrm{r},q} * \mathbf{w}_{\mathrm{t},q}$$

- Final composite image = sum of component images

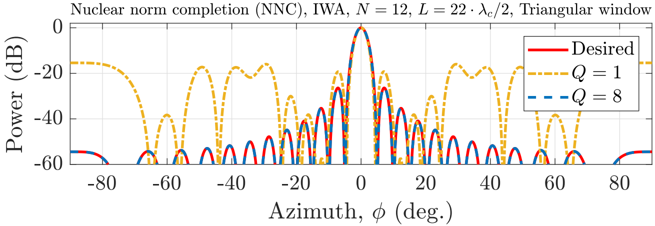
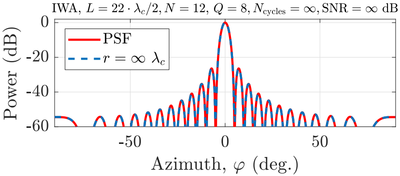
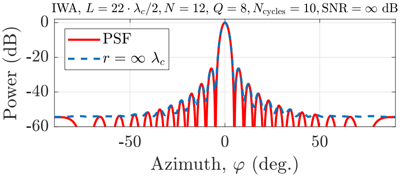


Figure: Point spread function of IWA. A single component ($Q = 1$) does not suppress the grating lobes of the sparse array. The desired PSF is achieved by $Q = 8$.

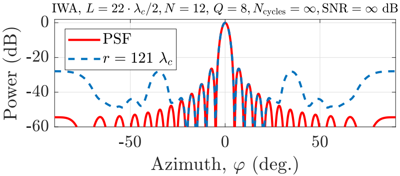
Wideband and near field point spread function



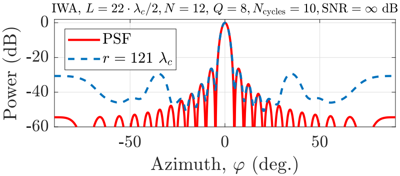
(a) Far field narrowband



(b) Far field, wideband



(c) Near field, narrowband



(d) Near field, wideband

Figure: Point spread function of IWA. In (a), the desired PSF is perfectly matched under far field narrowband conditions. In (b), the wideband signal smears out the nulls, but does not degrade side lobe levels or main lobe width. In (c) and (d), near field effects dominate and produce grating lobes.

Wideband and near field co-array

- Wideband $\rightarrow$ co-array scales linearly with frequency:

$$\mathcal{C}_{\text{wb}} = \bigcup_i \frac{f_i}{f_c} \mathcal{C}_c$$

- Near field $\rightarrow$ spatially varying co-array (single target):

$$d_{\Sigma,i} \approx (d_m + d_n) + \frac{1}{2r_k}(d_m^2 + d_n^2)(\sin\varphi_k + \sin\varphi)$$

- Co-array no longer independent of target range r_k, direction $\sin\varphi_k$ or array geometry[10]

[10] For more see: [Kozick and Kassam, 1993, Ahmad and Kassam, 2001, Ahmad et al., 2004, Coviello et al., 2012, He and Kassam, 2015]

SNR in coherent imaging (full phased array mode)

- Rx signal (single target, Rx noise, unit gain weights):

$$\tilde{s} = \sum_{n=1}^{N} (Ns + \xi_n) = N^2 s + \sum_{n=1}^{N} \xi_n$$

- Zero-mean, spatially white noise, and $\mathbb{E}[|s|^2] = P$:

$$\mathbb{E}[\xi_i \xi_j] = \begin{cases} 0, & \text{when } i \neq j \\ \sigma^2, & \text{otherwise.} \end{cases}$$

$$\tilde{\sigma}^2 = \mathbb{E}[|\sum_{n=1}^{N} \xi_n|^2] = N\sigma^2$$

$$\mathbb{E}[|\tilde{s}|^2] = N^4 P + \tilde{\sigma}^2$$

- Signal-to-noise ratio:

$$\text{SNR} = (\mathbb{E}[|\tilde{s}|^2] - \tilde{\sigma}^2)/\tilde{\sigma}^2 = N^3 P/\sigma^2 \propto 30 \log(N) \text{ dB}$$