Multi-Sensor Generalized Sequential Probability Ratio Test Using Level-Triggered Sampling

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Motivations

- Hypothesis testing
  - Multiple sensors: Diversity
  - Sequential test: Quick decision
- Applications:
  - Medical sensors: Remote diagnosis and health monitoring

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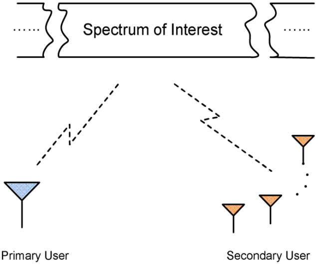
  - Multiple sensors: Diversity
  - Sequential test: Quick decision
- Applications:
  - Medical sensors: Remote diagnosis an health monitoring
  - Cognitive radio: Idle spectrum sensing



Outline

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Problem Statement

- Sequential hypothesis testing: Composite v.s. composite

$$\begin{array}{llll} \mathcal{H}_0: & Y_t \sim h_\gamma(y), & \gamma \in \Gamma, & t = 1, 2, \ldots \\ \mathcal{H}_1: & Y_t \sim f_\theta(y), & \theta \in \Theta, & t = 1, 2, \ldots \end{array}$$

  - Stopping time $\mathsf{T} : \{Y_1, Y_2, \ldots\} \to \mathbb{N}$
  - Decision function $\delta : \{Y_1, \ldots, Y_\mathsf{T}\} \to \{0, 1\}$

- Formulation:

$$\begin{array}{lll} \inf\limits_{\{\mathsf{T},\delta\}} & \mathbb{E}_x \mathsf{T}, & x \in \Gamma \cup \Theta \\ \text{subject to} & \sup\limits_\gamma \mathbb{P}_\gamma(\delta = 1) \le \alpha, & \sup\limits_\theta \mathbb{P}_\theta(\delta = 0) \le \beta \end{array}$$

$\Rightarrow$ Asymptotic optimality (as $\alpha, \beta \to 0$)

Problem Statement

- Define the feasible class of sequential tests:

$$\mathcal{T}(\alpha, \beta) \triangleq \left\{ \{\mathsf{T}, \delta\} : \sup_{\gamma} \mathbb{P}_{\gamma}(\delta = 1) \leq \alpha, \ \sup_{\theta} \mathbb{P}_{\theta}(\delta = 0) \leq \beta \right\}$$

Theorem (Li&Liu'14)

For any sequential test $\{\mathsf{T}, \delta\} \in \mathcal{T}(\alpha, \beta)$, its expected sample size is lower bounded by

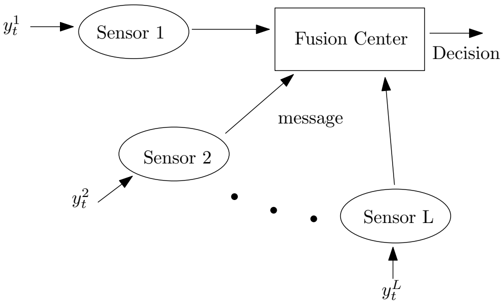
$$\mathbb{E}_{\gamma}(\mathsf{T}) \geq \frac{-\log \beta}{\inf_{\theta \in \Theta} D(h_{\gamma} || f_{\theta})} + o(-\log \beta)$$

$$\mathbb{E}_{\theta}(\mathsf{T}) \geq \frac{-\log \alpha}{\inf_{\gamma \in \Gamma} D(f_{\theta} || h_{\gamma})} + o(-\log \alpha)$$

Kullback-Leibler divergence: $D(f||h) = \mathbb{E}_f \left(\log \frac{f(X)}{h(X)} \right)$

Problem Statement

- Challenges:
  - Low battery at sensors
  - Limited wireless bandwidth
- Remedies:
  - Lower precision (**one-bit message**)
  - Lower frequency (**every** T_0 **steps, fixed or on average**)



$\Rightarrow$ **Can we devise a good test under this setup?**

Outline

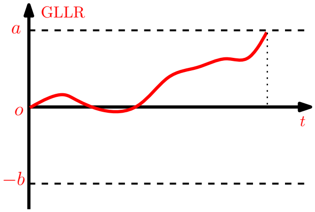
Generalized Sequential Probability Ratio Test

- Statistic: Generalized log-likelihood ratio (GLLR)

$$\mathrm{GLLR}(1,t) \triangleq \max_{\theta\in\Theta} \sum_{j=1}^{t} \log f_\theta(y_j^\ell) - \max_{\gamma\in\Gamma} \sum_{j=1}^{t} \log h_\gamma(y_j^\ell)$$

- Stopping rule: $\mathsf{T} \triangleq \inf\{t : \mathrm{GLLR}(1,t) \notin (-b,a)\}$
- Decision function: $\delta_\mathsf{T} \triangleq \mathbb{1}_{\{\mathrm{GLLR}(1,\mathsf{T})\geq a\}}$

$\Rightarrow$ Asymptotically optimal



Proposed: Level-Triggered Sampling

- **At sensors:**
  - When to transmit:

$$t_n \triangleq \inf\{t : \mathrm{GLLR}(t_{n-1}, t) \notin (-b, a)\}, \quad n = 1, 2, \ldots, \; t_0 = 0$$

  - What to transmit:

$$u_n \triangleq \begin{cases} +1, & \text{if } \mathrm{GLLR}(t_{n-1}, t_n) \geq a \\ -1, & \text{if } \mathrm{GLLR}(t_{n-1}, t_n) \leq -b \end{cases}$$

  - In brief: Repeat GSPRT → local decisions → binary messages

Proposed: Level-Triggered Sampling

- **At fusion center:**
  - Global statistic:

$$V_t \triangleq \sum_{\ell=1}^{L} \sum_{n=1}^{N_t^\ell} \left(a\, \mathbb{1}_{\{u_n^\ell = 1\}} - b\, \mathbb{1}_{\{u_n^\ell = -1\}} \right)$$

  - Stopping rule & decision rule:

$$\mathrm{T}_p \triangleq \inf\left\{t : V_t \notin (-B, A)\right\}, \quad \delta_p \triangleq \mathbb{1}_{\{V_{\mathrm{T}_p} \geq A\}}$$

  - In brief:

    Sum up the local boundary values, and make decision as the sum hits either global boundary.

Performance

Lemma (Expected Stopping Time)

In the asymptotic regime where $b, a \to \infty$ and $A/a, B/b \to \infty$, LTS-GSPRT yields the following expected sample size:

$$\mathbb{E}_{\gamma}\left(\mathsf{T}_p\right) \sim \frac{B}{\inf_{\theta} D\left(h_{\gamma} || f_{\theta}\right) L}, \quad \mathbb{E}_{\theta}\left(\mathsf{T}_p\right) \sim \frac{A}{\inf_{\gamma} D\left(f_{\theta} || h_{\gamma}\right) L}.$$

Lemma (Error Probabilities)

In the asymptotic regime where $b, a \to \infty$ and $A/a, B/b \to \infty$, the LTS-GSPRT yields the following type-I and type-II error probabilities:

$$\sup_{\gamma \in \Gamma} \log \mathbb{P}_{\gamma}(\delta_p = 1) \sim -A, \quad \sup_{\theta \in \Theta} \log \ \mathbb{P}_{\theta}\left(\delta_p = 0\right) \sim -B.$$

Performance

- Asymptotic performance:

$$\mathbb{E}_\gamma\left(\mathsf{T}_p\right) \sim \frac{-\log \beta}{\inf_{\theta\in\Theta} D\left(h_\gamma || f_\theta\right) L}, \quad \mathbb{E}_\theta\left(\mathsf{T}_p\right) \sim \frac{-\log \alpha}{\inf_{\gamma\in\Gamma} D\left(f_\theta || h_\gamma\right) L}$$

Theorem

The decentralized LTS-GSPRT $\{T_p, \delta_p\}$ is asymptotically optimal, in the asymptotic regime where $b, a \to \infty$ and $A/a, B/b \to \infty$:

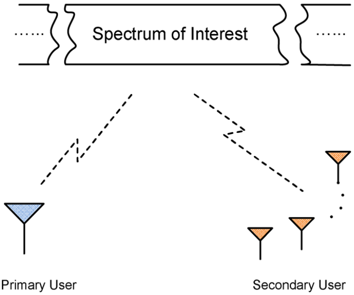
$$\mathbb{E}_x\left(\mathsf{T}_p\right) \sim \inf_{\{\mathsf{T},\delta\}\in\mathcal{T}(\alpha,\beta)} \mathbb{E}_x\left(\mathsf{T}\right), \qquad x \in \Gamma \cup \Theta.$$

Outline

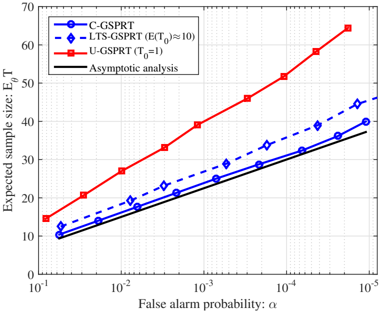
Application: Cooperative Spectrum Sensing

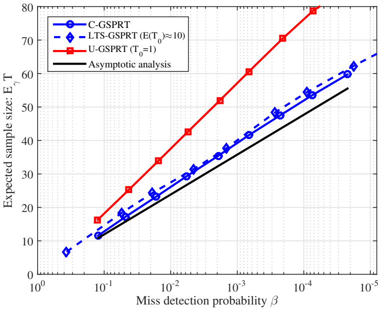
- Spectrum Sensing Model [Sergura&Wang'10]:

$$\mathcal{H}_0: \quad Y_t^\ell \sim \mathcal{N}(0,\gamma), \quad 0 < \gamma_0 \le \gamma \le \gamma_1, \quad \ell \in \mathcal{L}, \; t = 1,2,\ldots,$$
$$\mathcal{H}_1: \quad Y_t^\ell \sim \mathcal{N}(0,\theta), \quad \gamma_1 < \theta_0 \le \theta \le \theta_1 \quad \ell \in \mathcal{L}, \; t = 1,2,\ldots,$$

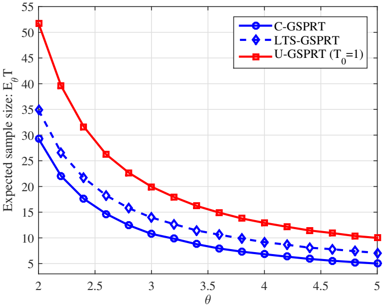


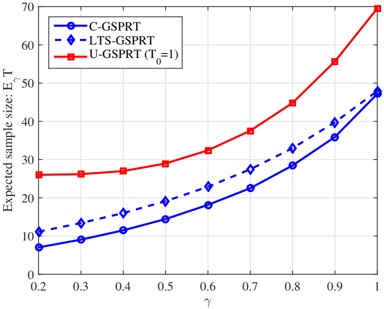
Results


C-GSPRT
LTS-GSPRT (E(T0)≈10)
U-GSPRT (T0=1)
Asymptotic analysis
Expected sample size: E_θ T
False alarm probability: α
70
60
50
40
30
20
10
0
10^-1
10^-2
10^-3
10^-4
10^-5

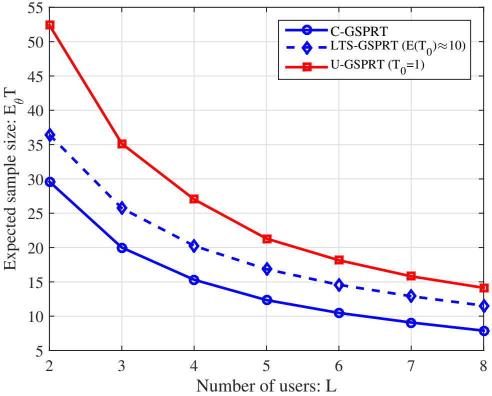

C-GSPRT
LTS-GSPRT (E(T0)≈10)
U-GSPRT (T0=1)
Asymptotic analysis
Expected sample size: E_γ T
Miss detection probability β
80
70
60
50
40
30
20
10
0
10^0
10^-1
10^-2
10^-3
10^-4
10^-5

Results


C-GSPRT
LTS-GSPRT
U-GSPRT (T0=1)
Expected sample size: Eθ T
θ
55
50
45
40
35
30
25
20
15
10
5
2
2.5
3
3.5
4
4.5
5


C-GSPRT
LTS-GSPRT
U-GSPRT (T0=1)
Expected sample size: Eγ T
γ
70
60
50
40
30
20
10
0
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Results


C-GSPRT
LTS-GSPRT (E(T0)≈10)
U-GSPRT (T0=1)
Expected sample size: EθT
Number of users: L
55
50
45
40
35
30
25
20
15
10
5
2
3
4
5
6
7
8

Q&A