Joint Composite Detection and Bayesian Estimation: A Neyman-Pearson Approach

Shang Li

Electrical Engineering Department
Columbia University

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Outline

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Joint Detection & Estimation

- Binary hypotheses:

$$\begin{array}{lll} \mathcal{H}_0: & \mathbf{y} \sim f_0\left(\mathbf{y}|\boldsymbol{\theta}\right), & \text{with } \boldsymbol{\theta} \sim \pi_0(\boldsymbol{\theta}), \\ \mathcal{H}_1: & \mathbf{y} \sim f_1\left(\mathbf{y}|\boldsymbol{\theta}\right), & \text{with } \boldsymbol{\theta} \sim \pi_1(\boldsymbol{\theta}), \end{array}$$

  - $f_i(\cdots|\boldsymbol{\theta}), i = 0, 1$: PDFs parameterized by $\boldsymbol{\theta}$
  - $\pi_i(\boldsymbol{\theta}), i = 0, 1$: prior on the parameters $\boldsymbol{\theta}$
- **Goals:**
  - Detection: decide between the competing hypotheses
  - Estimation: generate the estimate of parameters

Motivations

- Spectrum sensing in cognitive radio
  - $\boldsymbol{\theta}$ is the variance of received samples
  - Decide the presence of PU over a target spectrum
  - Estimate the noise (PU absent)/interference (PU present) level
- Power system state estimation
  - $\boldsymbol{\theta}$ is states of electric buses
  - Decide the presence of line fault
  - Estimate the states of electric buses under either decision

Conventional Approaches

- Generalized Likelihood Ratio Test:
  - δ : $\frac{f_1(\mathbf{y}|\hat{\boldsymbol{\theta}}_1)}{f_0(\mathbf{y}|\hat{\boldsymbol{\theta}}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$
  - $\hat{\boldsymbol{\theta}}$: $\hat{\boldsymbol{\theta}}_i = \arg\max_{\boldsymbol{\theta}} f_i(\mathbf{y}|\boldsymbol{\theta})\pi_i(\boldsymbol{\theta})$
- Likelihood Ratio Test + MMSE:
  - δ : $L(\mathbf{y}) \triangleq \frac{\int f_1(\mathbf{y}|\boldsymbol{\theta})\pi_1(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}}{\int f_0(\mathbf{y}|\boldsymbol{\theta})\pi_0(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma$
  - $\hat{\boldsymbol{\theta}}$: $\hat{\boldsymbol{\theta}}_i = \arg\min \mathbb{E}_i \left((\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2 \right) = \mathbb{E}_i \left(\boldsymbol{\theta}|\mathbf{y} \right)$

$\Rightarrow$ Not necessarily optimal in joint formulation

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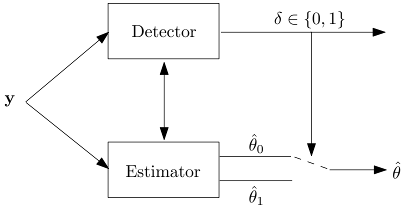
Formulation

- Coupling system [Middleton'68]:



- **How to characterize the system-wise performance?**
- **What is the optimal joint detector and estimator?**

Formulation

- Estimation metric:

$$\mathcal{C}\left(\hat{\boldsymbol{\theta}}_0, \hat{\boldsymbol{\theta}}_1, \delta\right) = \sum_{i=0}^{1} \mathbb{E}_i \left[c_{i,1} \|\hat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}\|^2 \mathbb{1}_{\{\delta=1\}} + c_{i,0} \|\hat{\boldsymbol{\theta}}_0 - \boldsymbol{\theta}\|^2 \mathbb{1}_{\{\delta=0\}} \right]$$

  - $c_{i,j}$'s are prescribed constants
  - $\mathbb{E}_i$'s are taken w.r.t. $g_i(\mathbf{y}, \boldsymbol{\theta}) = f_i(\mathbf{y}|\boldsymbol{\theta})\pi_i(\boldsymbol{\theta})$

- Detection metric:

$$\mathbb{P}_0 (\delta = 1) \quad \text{and} \quad \mathbb{P}_1 (\delta = 0)$$

Formulation

The Neyman-Pearson Type of Formulation

$$\begin{aligned} \text{minimize}_{\hat{\theta}_0, \hat{\theta}_1, \delta} \quad & \mathcal{C}\left(\hat{\boldsymbol{\theta}}_0, \hat{\boldsymbol{\theta}}_1, \delta\right) \\ \text{subject to} \quad & \mathbb{P}_0\left(\delta = 1\right) \leq \alpha, \quad \mathbb{P}_1\left(\delta = 0\right) \leq \beta. \end{aligned}$$

Insights:

– Extreme case: $\beta = \beta^{\mathsf{NP}}(\alpha)$, δ is fixed as LRT:

$$\mathcal{C}^\star(\delta_{\mathsf{LRT}}) = \text{minimize}_{\hat{\theta}_0, \hat{\theta}_1} \; \mathcal{C}\left(\hat{\boldsymbol{\theta}}_0, \hat{\boldsymbol{\theta}}_1, \delta_{\mathsf{LRT}}\right)$$

– Tradeoff: $\beta > \beta^{\mathsf{NP}}(\alpha)$ improves the estimation cost $\mathcal{C}^\star \leq \mathcal{C}^\star(\delta_{\mathsf{LRT}})$

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Main Results

Optimal Joint Esitmators

$$\hat{\boldsymbol{\theta}}_i^\star = \frac{c_{1,i}L(\mathbf{y})}{c_{1,i}L(\mathbf{y}) + c_{0,i}}\mathbb{E}_1\left(\boldsymbol{\theta}|\mathbf{y}\right) + \frac{c_{0,i}}{c_{1,i}L(\mathbf{y}) + c_{0,i}}\mathbb{E}_0\left(\boldsymbol{\theta}|\mathbf{y}\right), \quad i = 0, 1.$$

Insights:

- Adaptive to the true sample distribution:

$$\begin{aligned} &\text{under } \mathcal{H}_0: \quad L(\mathbf{y}) \downarrow \quad \Rightarrow \quad \hat{\boldsymbol{\theta}}^\star \to \mathbb{E}_0(\boldsymbol{\theta}|\mathbf{y}) \\ &\text{under } \mathcal{H}_1: \quad L(\mathbf{y}) \uparrow \quad \Rightarrow \quad \hat{\boldsymbol{\theta}}^\star \to \mathbb{E}_1(\boldsymbol{\theta}|\mathbf{y}) \end{aligned}$$

$\mathbb{E}_i(\boldsymbol{\theta}|\mathbf{y})$'s are standard MMSE under $\mathcal{H}_i$

Main Results

Joint Detector

$$[\lambda_1^\star - D_{1,1}(\mathbf{y}) + D_{1,0}(\mathbf{y})]\, L(\mathbf{y}) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda_0^\star - D_{0,0}(\mathbf{y}) + D_{0,1}(\mathbf{y})$$

with $D_{i,j}(\mathbf{y}) \triangleq c_{i,j}\mathbb{E}_i\left(\|\hat{\boldsymbol{\theta}}_j^\star - \boldsymbol{\theta}\|^2 \middle| \mathbf{y} \right)$ is the posterior estimation cost for decision $\delta = j$ under $\mathcal{H}_i$.

Insights:

– Coupled with estimation

$$D_{1,1}(\mathbf{y}) \downarrow \Rightarrow \delta = 1 \text{ more likely to be correct}$$
$$D_{1,0}(\mathbf{y}) \uparrow \Rightarrow \delta = 0 \text{ more likely to be wrong}$$

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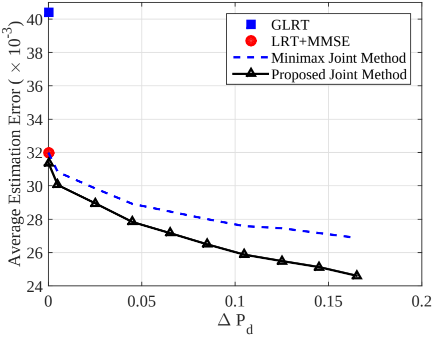
Results

- The testing formulation for the spectrum sensing problem [Font-Segura&Wang'10]:

$$\mathcal{H}_0 : \mathbf{y} \sim \mathcal{N}\left(\mathbf{0}, \sigma_0^2 \mathbf{I}_n\right), \text{with } \sigma_0^2 \sim \pi_0(\sigma_0^2) = \chi^{-2}(\nu_0, \ell_0)$$
$$\mathcal{H}_1 : \mathbf{y} \sim \mathcal{N}\left(\mathbf{0}, \sigma_1^2 \mathbf{I}_n\right), \text{with } \sigma_1^2 \sim \pi_1(\sigma_1^2) = \chi^{-2}(\nu_1, \ell_1)$$

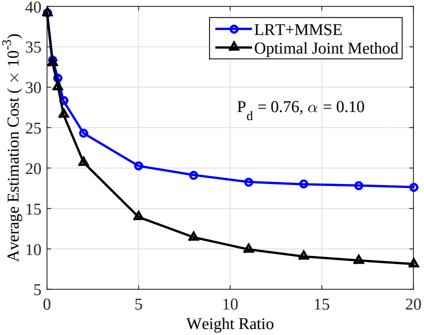
  - $\sigma_i^2, i = 0, 1$ is the variance of the received samples, with prior depending on the presence of primary user (PU)
  - Decide the presence of PU over a target spectrum
  - Estimate the noise (PU absent)/interference (PU present) level

Results



— Tradeoff between detection and estimation performances

Results



– Improving the estimation accuracy with adaptive estimators

Q&A