An Improved DOA Estimator Based On Partial Relaxation Approach

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Motivation

- Wide application of DOA estimation

- Multiple families of DOA estimators:
  - Maximum likelihood estimators
  - Subspace-based estimators
  - ...

- Proposal of a DOA estimator under the Partial Relaxation approach
  - Closely related to conventional DOA estimators
  - Efficient implementation for updating eigenvalues

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Signal Model

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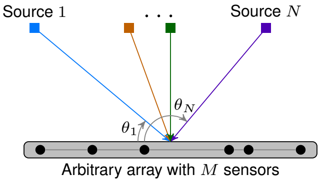
Multiple Snapshots Model

$$\mathbf{X} = \mathbf{A}(\boldsymbol{\theta})\mathbf{S} + \mathbf{N}$$



- T : Number of available snapshots
- $\mathbf{X} \in \mathbb{C}^{M \times T}$: Received signal matrix
- $\mathbf{S} \in \mathbb{C}^{N \times T}$: Source signal matrix
- $\mathbf{N} \in \mathbb{C}^{M \times T}$: Sensor noise matrix
- $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), ..., \mathbf{a}(\theta_N)] \in \mathbb{C}^{M \times N}$: Steering matrix

Array Manifold

$$\mathcal{A}_N = \left\{ \mathbf{A} \in \mathbb{C}^{M \times N} | \mathbf{A} = [\mathbf{a}(\vartheta_1), \dots, \mathbf{a}(\vartheta_N)] \text{ with } 0 \leq \vartheta_1 < \dots < \vartheta_N < 180^\circ \right\}$$

Signal Model

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Covariance Matrix

$$\mathbf{R} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}_M$$

- $\mathbf{R} = \mathbb{E}\left\{\mathbf{x}(t)\mathbf{x}(t)^H\right\} \in \mathbb{C}^{M\times M}$: Covariance matrix of the received signal
- $\mathbf{R}_s = \mathbb{E}\left\{\mathbf{s}(t)\mathbf{s}(t)^H\right\} \in \mathbb{C}^{N\times N}$: Covariance matrix of the transmitted signal
- σ_n^2: Noise power at the sensors

Sample Covariance Matrix

$$\hat{\mathbf{R}} = \frac{1}{T}\mathbf{X}\mathbf{X}^H = \hat{\mathbf{U}}_s\hat{\mathbf{\Lambda}}_s\hat{\mathbf{U}}_s^H + \hat{\mathbf{U}}_n\hat{\mathbf{\Lambda}}_n\hat{\mathbf{U}}_n^H$$

- Signal subspace spanned by $\hat{\mathbf{U}}_s$
- N largest eigenvalues $\left\{\hat{\lambda}_1, \dots, \hat{\lambda}_N\right\}$ of $\hat{\mathbf{R}}$ are contained in $\hat{\mathbf{\Lambda}}_s$

- Noise subspace spanned by $\hat{\mathbf{U}}_n$
- $(M - N)$ smallest eigenvalues $\left\{\hat{\lambda}_{N+1}, \dots, \hat{\lambda}_M\right\}$ of $\hat{\mathbf{R}}$ are contained in $\hat{\mathbf{\Lambda}}_n$

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Partial Relaxation Approach

Revision of Conventional DOA Estimators

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General Formulation

$$\left\{\hat{\mathbf{A}}\right\} = \arg\min_{\mathbf{A}\in\mathcal{A}_N} f\left(\mathbf{A}\right)$$

Remarks

- $\mathcal{A}_N$: Highly structured and non-convex set
- $f(\cdot)$: Generally non-convex function with multiple local minima
- Highly computational cost to obtain the global minimum

Example: Deterministic Maximum Likelihood (DML) Estimator

$$\left\{\hat{\mathbf{A}}_{\text{DML}}\right\} = \arg\min_{\mathbf{A}\in\mathcal{A}_N} \text{tr}\left\{\left(\mathbf{I}_M - \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\right)\hat{\mathbf{R}}\right\}$$

Partial Relaxation Approach

Revision of Conventional DOA Estimators

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General Formulation

$$\left\{\hat{\mathbf{A}}\right\} = \arg\min_{\mathbf{A}\in\mathcal{A}_N} f\left(\mathbf{A}\right)$$

Remarks

- $\mathcal{A}_N$: Highly structured and non-convex set
- $f(\cdot)$: Generally non-convex function with multiple local minima
- Highly computational cost to obtain the global minimum

Example: Covariance Fitting (CF) Estimator

$$\left\{\hat{\mathbf{A}}_{\text{CF}}\right\} = \arg\min_{\mathbf{A}\in\mathcal{A}_N} \min_{\mathbf{R_s}\succeq\mathbf{0}} \left|\left|\hat{\mathbf{R}} - \mathbf{A}\mathbf{R}_s\mathbf{A}^H\right|\right|_F^2$$

Partial Relaxation Approach

Revision of Conventional DOA Estimators

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General Formulation

$$\left\{\hat{\mathbf{A}}\right\} = \arg\min_{\mathbf{A}\in\mathcal{A}_N} f\left(\mathbf{A}\right)$$

Remarks

- $\mathcal{A}_N$: Highly structured and non-convex set
- $f(\cdot)$: Generally non-convex function with multiple local minima
- Highly computational cost to obtain the global minimum

Objective: Find a suboptimal solution without substantial performance degradation

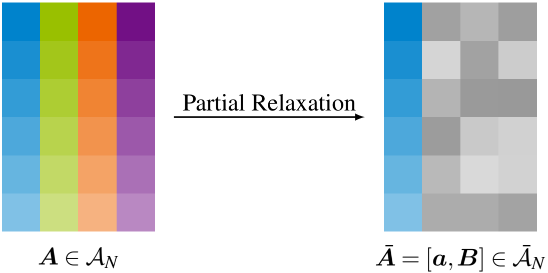
Partial Relaxation Approach
Concept

General Formulation of Conventional Estimators

$$\left\{\hat{\mathbf{A}}\right\} = \arg\min_{\mathbf{A}\in\mathcal{A}_N} f\left(\mathbf{A}\right)$$

Relaxed Array Manifold

$$\bar{\mathcal{A}}_N = \left\{\mathbf{A} \in \mathbb{C}^{M\times N} \middle| \mathbf{A} = \left[\mathbf{a}(\vartheta), \mathbf{B}\right], \mathbf{a}(\vartheta) \in \mathcal{A}_1, \mathbf{B} \in \mathbb{C}^{M\times(N-1)}\right\}$$



Partial Relaxation Approach

Concept

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General Formulation of Conventional Estimators

$$\left\{\hat{\mathbf{A}}\right\} = \underset{\mathbf{A}\in\mathcal{A}_N}{\arg\min}\, f\left(\mathbf{A}\right)$$

Relaxed Array Manifold

$$\bar{\mathcal{A}}_N = \left\{\mathbf{A} \in \mathbb{C}^{M\times N} \middle| \mathbf{A} = \left[\mathbf{a}(\vartheta), \mathbf{B}\right], \mathbf{a}(\vartheta) \in \mathcal{A}_1, \mathbf{B} \in \mathbb{C}^{M\times(N-1)}\right\}$$

Formulation of the Partial Relaxation (PR) Approach

$$\left\{\hat{\mathbf{a}}_{\mathrm{PR}}\right\} = {}^{N}\underset{\mathbf{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\mathbf{B}\in\mathbb{C}^{M\times(N-1)}} f\left(\left[\mathbf{a}, \mathbf{B}\right]\right)$$

- Relax the manifold structure of the signals from interfering directions
- Grid-search for N-deepest local minima to obtain the estimated DOAs

Partial Relaxation Approach

Proposed Estimators

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Formulation of PR-Constrained Covariance Fitting (PR-CCF)

$$\left\{\hat{\mathbf{a}}_{\text{PR-CCF}}\right\} = {}^{N}\underset{\mathbf{a}\in\mathcal{A}_1}{\arg\min}\; \min_{\sigma_s^2\geq 0,\mathbf{E}} \left|\left|\hat{\mathbf{R}} - \sigma_s^2\mathbf{a}\mathbf{a}^H - \mathbf{E}\mathbf{E}^H\right|\right|_F^2$$

$$\text{subject to } \hat{\mathbf{R}} - \sigma_s^2\mathbf{a}\mathbf{a}^H - \mathbf{E}\mathbf{E}^H \succeq \mathbf{0}$$

$$\text{rank}(\mathbf{E}) \leq N-1$$

PR-CCF Estimator

$$\left\{\hat{\mathbf{a}}_{\text{PR-CCF}}\right\} = {}^{N}\underset{\mathbf{a}\in\mathcal{A}_1}{\arg\min}\; \sum_{k=N}^{M}\lambda_k^2\left(\hat{\mathbf{R}} - \frac{1}{\mathbf{a}^H\hat{\mathbf{R}}^{-1}\mathbf{a}}\mathbf{a}\mathbf{a}^H\right)$$

Remarks

- Not applicable if $\hat{\mathbf{R}}$ is singular
- Eigenvalues are extensively required

Partial Relaxation Approach

Proposed Estimators

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PR-Unconstrained Covariance Fitting (PR-UCF)

$$\left\{\hat{\mathbf{a}}_{\text{PR-UCF}}\right\} = {}^{N}\underset{\mathbf{a}\in\mathcal{A}_1}{\arg\min}\ \min_{\sigma_s^2\geq 0,\mathbf{E}} \left|\left|\hat{\mathbf{R}} - \sigma_s^2\mathbf{a}\mathbf{a}^H - \mathbf{E}\mathbf{E}^H\right|\right|_F^2$$

$$\text{subject to rank}(\mathbf{E}) \leq N-1$$

Equivalent formulation of the inner optimization

$$\min_{\sigma_s^2\geq 0} \sum_{k=N}^{M} \lambda_k^2\left(\hat{\mathbf{R}} - \sigma_s^2\mathbf{a}\mathbf{a}^H\right)$$

- No closed-form solution for the minimizer $\hat{\sigma}_{s,\text{U}}^2$
- $\lambda_k^2\left(\hat{\mathbf{R}} - \sigma_s^2\mathbf{a}\mathbf{a}^H\right)$ is continuously differentiable with respect to σ_s^2

Minimization by Bisection Search or Newton's Method possible

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Computational Aspects

Core Numerical Problem

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Objective: Efficient computation of the eigenvalue decomposition

$$\mathbf{D} - \rho \mathbf{z}\mathbf{z}^H = \bar{\mathbf{U}}\bar{\mathbf{D}}\bar{\mathbf{U}}^H$$

- $\mathbf{D} = \text{diag}(d_1, \ldots, d_K) \in \mathbb{R}^{K \times K}$ with $d_1 > \ldots > d_K$
- $\rho > 0$
- $\mathbf{z} = [z_1, \ldots, z_K]^T \in \mathbb{C}^{K \times 1}$ has no zero component

- $\bar{\mathbf{D}} = \text{diag}(\bar{d}_1, \ldots, \bar{d}_K) \in \mathbb{R}^{K \times K}$ with $\bar{d}_1 > \ldots > \bar{d}_K$
- $\bar{\mathbf{U}} = [\bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_K] \in \mathbb{C}^{K \times K}$ contains the normalized eigenvectors $\bar{\mathbf{u}}_k$ associated with the eigenvalues $\bar{d}_k$

Interlacing Property

a) The modified eigenvalues $\bar{d}_k$ satisfy $h(\bar{d}_k) = 0$ where the secular function $h(x)$ is defined as:

$$h(x) = 1 - \rho \mathbf{z}^H (\mathbf{D} - x\mathbf{I})^{-1} \mathbf{z}$$
$$= 1 - \rho \sum_{k=1}^{K} \frac{|z_k|^2}{d_k - x}$$

b) The modified eigenvalues $\bar{d}_k$ down-interlace with the initial eigenvalues d_k

$$d_1 > \bar{d}_1 > d_2 > \bar{d}_2 > ... > d_K > \bar{d}_K$$

Objective: Determine $\bar{d}_k$ which satisfies $h(\bar{d}_k) = 0$

Computational Aspects
Iterative Algorithm

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Rewriting the secular function for determining $\bar{d}_k$

$$
\begin{aligned}
0 &= 1 - \sum_{i=1}^{k} \frac{|z_i|^2}{d_i - x} - \sum_{i=k+1}^{K} \frac{|z_i|^2}{d_i - x} \\
\iff & \sum_{i=1}^{k} \frac{|z_i|^2}{d_i - x} = 1 - \sum_{i=k+1}^{K} \frac{|z_i|^2}{d_i - x} \\
\iff & -\psi_k(x) = 1 + \phi_k(x)
\end{aligned}
$$

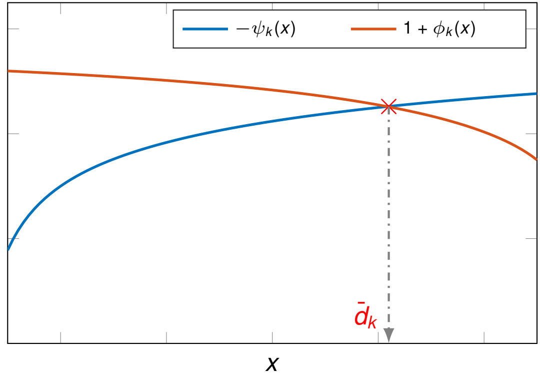
Idea: Approximate ψ_k and ϕ_k with a rational function of first degree

$$
R_{k;p,q}(x) = \begin{cases} p + \dfrac{q}{d_{k+1} - x} & \text{if } 0 \leq k \leq K-1 \\ 0 & \text{if } k = K, \end{cases}
$$

Computational Aspects

Iterative Algorithm

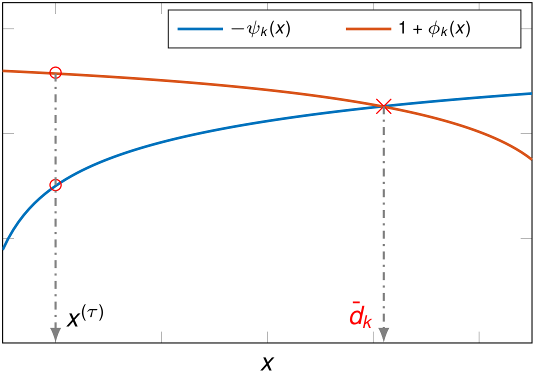
Rational Approximation



Computational Aspects

Iterative Algorithm

Rational Approximation



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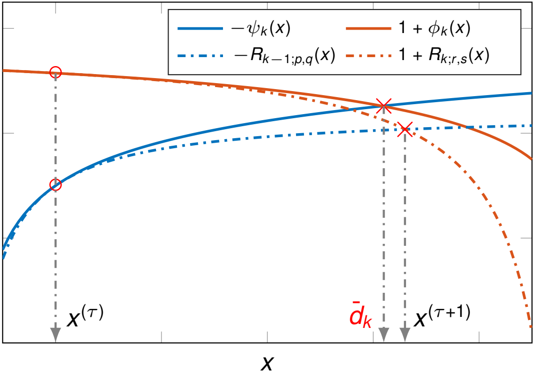
Rational Approximation



Computational Aspects

Iterative Algorithm

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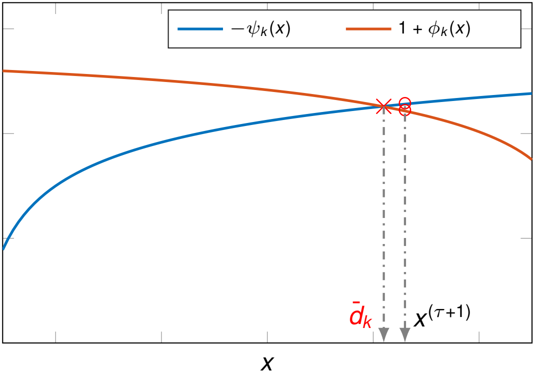
Rational Approximation



- Closed-form update
- Quadratic rate of convergence

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Simulation Results
Influence of SNR

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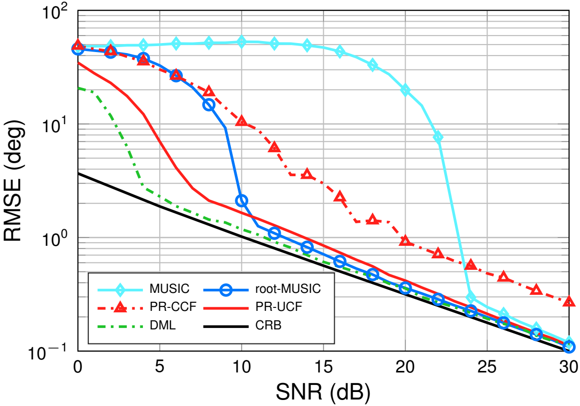
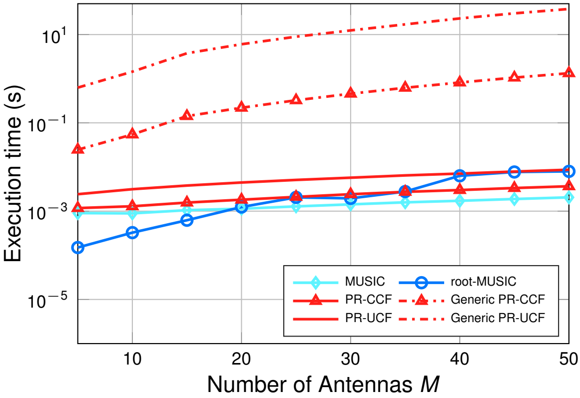
$M = 10, \theta = [45^\circ, 50^\circ]^T, T = 8$



Simulation Results

Execution Time

$M = 10, \theta = [45^\circ, 50^\circ]^T, T = 100, N_G = 1800$



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Conclusions and Outlook

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Conclusions

- Structure of interfering directions is relaxed
- Proposed estimator based on the covariance fitting problem
- Improved non-asymptotic behavior without exploiting any special structure of the sensor array
- Efficient implementation using rank-one update

Outlook

- Statistical properties of the proposed DOA estimator
- Generalization to multidimensional parameter estimation

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Thank you for your attention!

Appendix

Numerical Solution for PR-UCF using Bisection Search

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Define

$$g(\sigma_s^2) = \sum_{k=N}^{M} \bar{\lambda}_k(\sigma_s^2) = \sum_{k=N}^{M} \lambda_k^2 \left(\hat{\mathbf{R}} - \sigma_s^2 \mathbf{a}\mathbf{a}^H\right)$$

Asymptotic analysis of $g'(\sigma_s^2)$

$$g'(\sigma_s^2) = -\sum_{k=N}^{M} \frac{2\bar{\lambda}_k(\sigma_s^2)}{\sigma_s^4\ \mathbf{a}^H \left(\hat{\mathbf{R}} - \bar{\lambda}_k(\sigma_s^2)\mathbf{I}_M\right)^{-2} \mathbf{a}}$$

- If $\sigma_s^2 \to 0 \Rightarrow g'(\sigma_s^2) < 0$
- If $\sigma_{s,0}^2 \to \infty$:

$$g\left(\sigma_s^2\right) \approx \sigma_s^4 \, ||\mathbf{a}||_2^4$$
$$g'(\sigma_s^2) \to +\infty$$

Minimization by finding an interval where $g'(\sigma_s^2)$ changes sign and performing bisection search

Appendix

Numerical Solution for PR-UCF using Newton's Method

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Define

$$A = \sum_{j=1}^{N} \frac{\left|z_j\right|^2}{\left(\hat{\lambda}_j - \bar{\lambda}_k\left(\sigma_s^2\right)\right)^2}$$

$$B = \sum_{j=1}^{N} \frac{\left|z_j\right|^2}{\left(\hat{\lambda}_j - \bar{\lambda}_k\left(\sigma_s^2\right)\right)^3}$$

Second derivative of $g(\sigma_s^2) = \sum_{k=N}^{M} \bar{\lambda}_k^2\left(\sigma_s^2\right)$

$$g''(\sigma_s^2) = \sum_{k=N}^{M} \frac{4\sigma_s^2\bar{\lambda}_k\left(\sigma_s^2\right)A^2 - 4\bar{\lambda}_k\left(\sigma_s^2\right)B + A}{\sigma_s^8 A^3}$$

Appendix

Eigenvalue Computation

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Algorithm 1 Determining the k-th eigenvalue $\bar{d}_k$

1: **Initial.**: Iteration index $\tau = 0$, initial point $x^{(0)} \in (d_{k+1}, d_k)$, tolerance $\epsilon = 10^{-9}$

2: **repeat**

3: Approximate $\psi_k(x)$ by determining the parameters p and q such that:

$$R_{k-1;p,q}(x^{(\tau)}) = \psi_k(x^{(\tau)}) \text{ and } R'_{k-1;p,q}(x^{(\tau)}) = \psi'_k(x^{(\tau)})$$

4: Approximate $\phi_k(x)$ by determining the parameters r and s such that:

$$R_{k;r,s}(x^{(\tau)}) = \phi_k(x^{(\tau)}) \text{ and } R'_{k;r,s}(x^{(\tau)}) = \phi'_k(x^{(\tau)})$$

5: Determine $x^{(\tau+1)} \in (d_{k+1}, d_k)$ which satisfies:

$$-R_{k-1;p,q}(x^{(\tau+1)}) = 1 + R_{k;r,s}(x^{(\tau+1)})$$

6: $\tau \leftarrow \tau + 1$

7: **until** $\left| x^{(\tau+1)} - x^{(\tau)} \right| < \epsilon$

8: **return** $\bar{d}_k = x^{(\tau+1)}$

Appendix

Eigenvalue Computation

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Remarks

- Eigenvalues of the previous direction are reused for initializations
- Reduced execution time by using properties of the trace operator

Application to PR-UCF

$$\bar{\lambda}_k(\sigma_s^2) = \lambda_k\left(\hat{\mathbf{R}} - \sigma_s^2 \mathbf{a}\mathbf{a}^H\right)$$

$$g'\left(\sigma_s^2\right) = -2\mathbf{a}^H\hat{\mathbf{R}}\mathbf{a} + 2\sigma_s^2 \left|\left|\mathbf{a}\right|\right|_2^4 + \sum_{k=1}^{N-1} \frac{2\bar{\lambda}_k\left(\sigma_s^2\right)}{\sigma_{s,0}^4 \mathbf{a}^H\left(\hat{\mathbf{R}} - \bar{\lambda}_k(\sigma_{s,0}^2)\mathbf{I}_M\right)^{-2}\mathbf{a}}$$

Appendix
Eigenvalue Computation

Remarks

- Eigenvalues of the previous direction are reused for initializations
- Reduced execution time by using properties of the trace operator

Application to PR-UCF

$$\bar{\lambda}_k(\sigma_s^2) = \lambda_k\left(\hat{\mathbf{\Lambda}} - \sigma_s^2 \mathbf{z}\mathbf{z}^H\right)$$

$$g'\left(\sigma_s^2\right) = -2\mathbf{z}^H\hat{\mathbf{\Lambda}}\mathbf{z} + 2\sigma_s^2 \left|\left|\mathbf{z}\right|\right|_2^4 + \sum_{k=1}^{N-1} \frac{2\bar{\lambda}_k\left(\sigma_s^2\right)}{\sigma_s^4 \sum_{j=1}^{M} \frac{\left|z_j\right|^2}{\left(\hat{\lambda}_j - \bar{\lambda}_k\left(\sigma_s^2\right)\right)^2}}$$

with $\hat{\mathbf{R}} = \hat{\mathbf{U}}\hat{\mathbf{\Lambda}}\hat{\mathbf{U}}^H$, $\mathbf{z} = \hat{\mathbf{U}}^H\mathbf{a}$

Appendix
Complexity Summary

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Total computational complexity (including overhead)

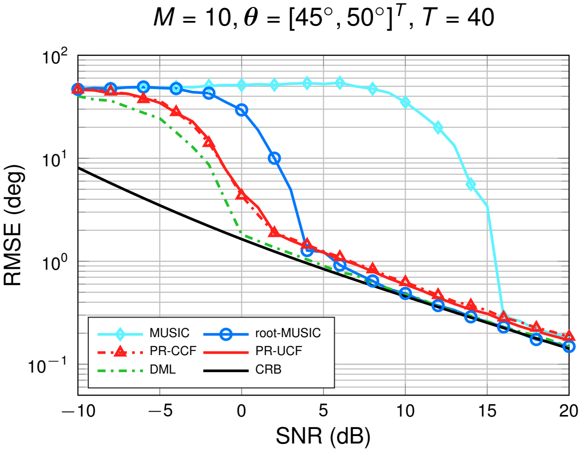
Estimator	**Generic**	**Rank-one Update**
PR-CCF	$O(M^3 N_G)$	$O(M^2 N_G)$
PR-UCF	$O(M^3 N_G N_I)$	$O(M^2 N_G N_I)$
MUSIC	$O(MNN_G)$	

Table: Complexity for computing the null-spectra

- M : Number of sensors
- N : Number of sources
- N_G : Number of look-directions of the complete angle-of-view
- N_I : Number of bisection steps

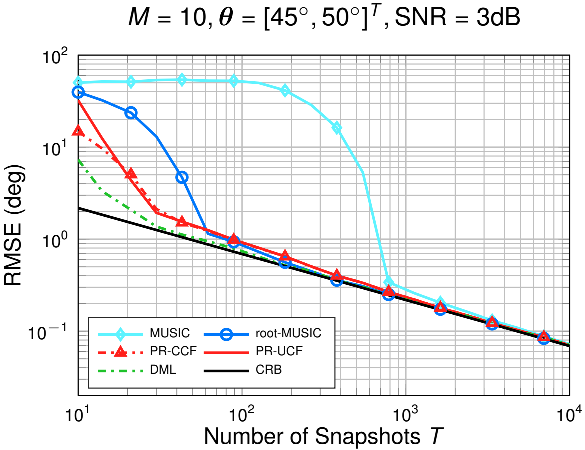
Appendix

Influence of SNR



Appendix

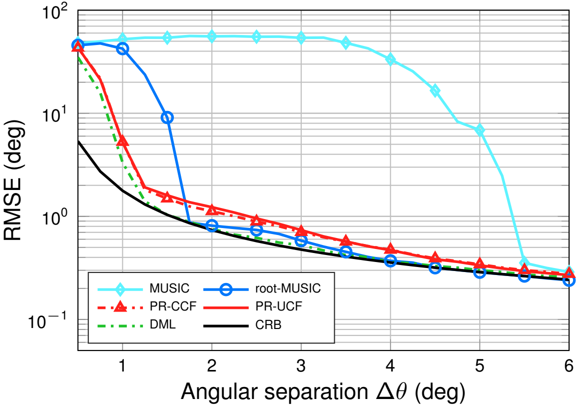
Influence of Number of Snapshots



Appendix

Influence of Angular Separation

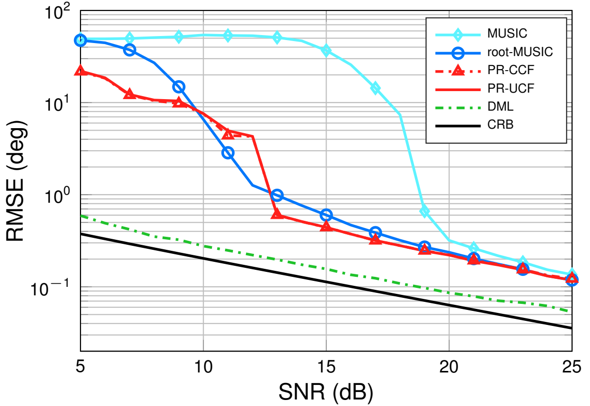
$M = 10, \boldsymbol{\theta} = [45^\circ, 45^\circ + \Delta\theta]^T$, SNR = 10dB, $T = 100$



Appendix

Correlated Source Signals

$M = 10, \boldsymbol{\theta} = [45^\circ, 50^\circ]^T, \rho = 0.95, T = 200$



Appendix
Bisection Method on PR-UCF

$M = 10, \boldsymbol{\theta} = [45^\circ, 50^\circ]^T, T = 100, N_G = 1800$