Source Separation in the Presence of Side Information: Necessary and Sufficient Conditions for Reliable De-mixing

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Introduction

Research Background

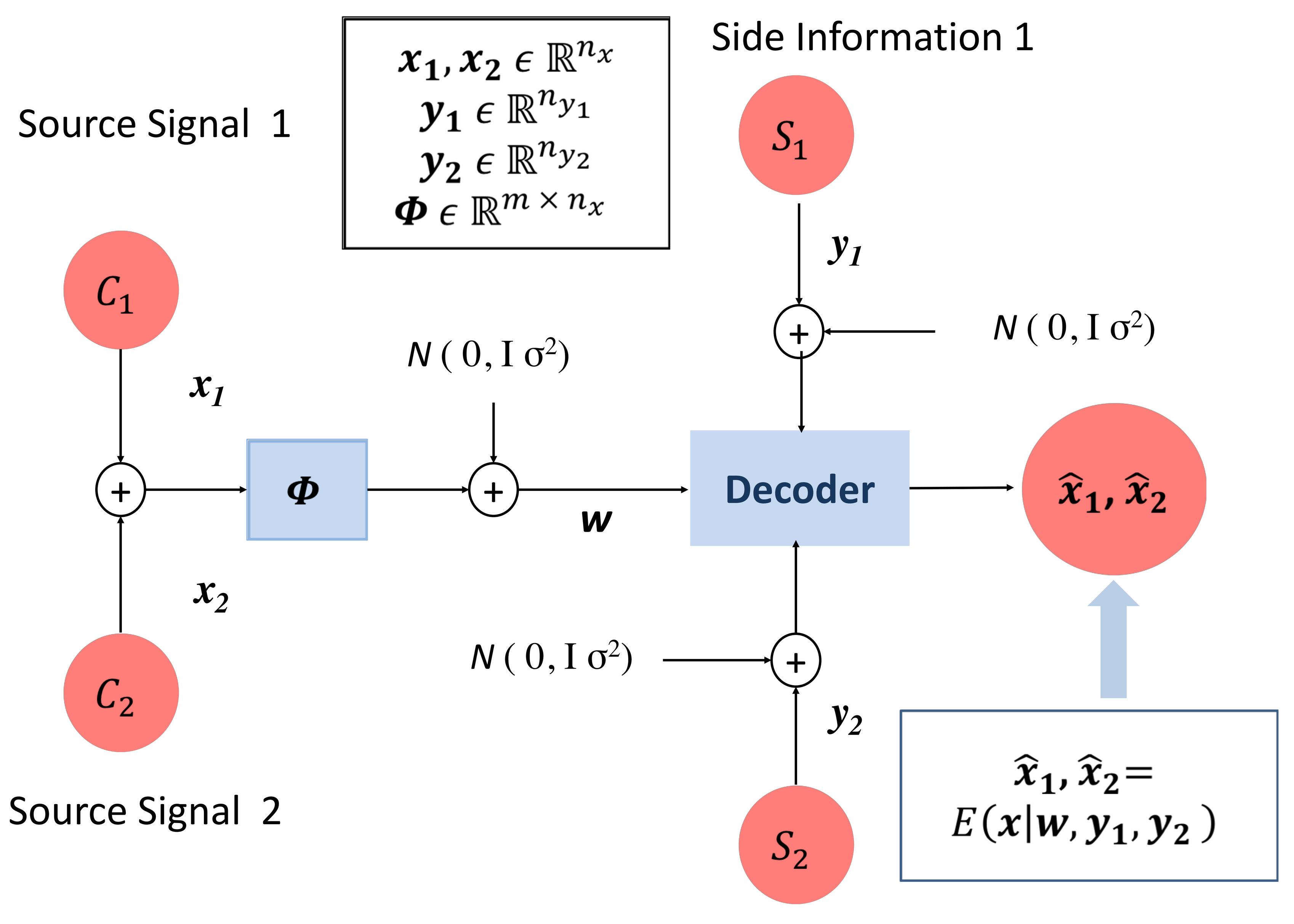
- Source Separation with Side information [1]
- Compressive Sensing with Side information [2]
- Compressive Sensing with Gaussian Mixture model (GMM) [3]

Contribution

- Studying the source separation problem in the presence of side information
- Providing the necessary and sufficient conditions for the reliable separation

Model

Source Separation with Side information



We will be assuming that $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ are statistically independent drawn from joint GMM, characterized by underlying class labels (C_1, S_1) and (C_2, S_2), obey the joint probability density function (pdf):

$$p(\boldsymbol{x_1}, \boldsymbol{y_1} \mid C_1 = j_1, S_1 = k_1) \sim N(\boldsymbol{\mu}_{x_1 y_1}^{(j_1 k_1)}, \overline{\boldsymbol{\Sigma}}_{x_1 y_1}^{(j_1 k_1)})$$

$$p(\boldsymbol{x_2}, \boldsymbol{y_2} \mid C_2 = j_2, S_2 = k_2) \sim N(\boldsymbol{\mu}_{x_2 y_2}^{(j_2 k_2)}, \overline{\boldsymbol{\Sigma}}_{x_2 y_2}^{(j_2 k_2)})$$

where $\boldsymbol{\mu}_{x_1 y_1}^{(j_1 k_1)}$, $\boldsymbol{\mu}_{x_2 y_2}^{(j_2 k_2)}$ and $\overline{\boldsymbol{\Sigma}}_{x_1 y_1}^{(j_1 k_1)}, \overline{\boldsymbol{\Sigma}}_{x_2 y_2}^{(j_2 k_2)}$ are the mean and covariance matrices and $\overline{\boldsymbol{\Sigma}}_{x_1 y_1}^{(j_1 k_1)} = \begin{pmatrix} \boldsymbol{\Sigma}_{x_1}^{(j_1 k_1)} & \boldsymbol{\Sigma}_{x_1 y_1}^{(j_1 k_1)} \\ \boldsymbol{\Sigma}_{y_1 x_1}^{(j_1 k_1)} & \boldsymbol{\Sigma}_{y_1}^{(j_1 k_1)} \end{pmatrix}$, $\overline{\boldsymbol{\Sigma}}_{x_2 y_2}^{(j_2 k_2)} = \begin{pmatrix} \boldsymbol{\Sigma}_{x_2}^{(j_2 k_2)} & \boldsymbol{\Sigma}_{x_2 y_2}^{(j_2 k_2)} \\ \boldsymbol{\Sigma}_{y_2 x_2}^{(j_2 k_2)} & \boldsymbol{\Sigma}_{y_2}^{(j_2 k_2)} \end{pmatrix}$. The covariance matrices are assumed to be low-rank where we denote such ranks by $r_{x_1 y_1}^{(j_1 k_1)} = \text{rank}\left(\overline{\boldsymbol{\Sigma}}_{x_1 y_1}^{(j_1 k_1)}\right)$, $r_{x_2 y_2}^{(j_2 k_2)} = \text{rank}\left(\overline{\boldsymbol{\Sigma}}_{x_2 y_2}^{(j_2 k_2)}\right)$, $r_{y_1}^{(j_1 k_1)} = \text{rank}\left(\boldsymbol{\Sigma}_{y_1}^{(j_1 k_1)}\right)$, $r_{y_2}^{(j_2 k_2)} = \text{rank}\left(\boldsymbol{\Sigma}_{y_2}^{(j_2 k_2)}\right)$. We use the MMSE as the performance measure denoted by $\text{MMSE}(\sigma^2) = E[||\boldsymbol{x} - E(\boldsymbol{x}|\boldsymbol{w}, \boldsymbol{y_1}, \boldsymbol{y_2})||^2]$ where $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{pmatrix}$.

Analysis

Necessary and sufficient condition for reliable separation

$\lim_{\sigma^2 \to 0} MMSE(\sigma^2) = 0 \implies m \geq r_{x_1 y_1}^{(j_1 k_1)} - r_{x_2 y_2}^{(j_2 k_2)} - r_{y_1}^{(j_1 k_1)} - r_{y_2}^{(j_2 k_2)}$ and $D^{(j_1 k_1, j_2 k_2)} = 0$ for $\forall \, (j_1 k_1, j_2 k_2) \in \mathcal{L}$, where $\mathcal{L}$ represents the set of probable labels and

$$D^{(j_1 k_1, j_2 k_2)} = \dim\left(\text{Im}\left(\boldsymbol{\Sigma}_{x_1|y_1}^{(j_1 k_1)}\right) \cap \text{Im}\left(\boldsymbol{\Sigma}_{x_2|y_2}^{(j_2 k_2)}\right)\right)$$

$$\boldsymbol{\Sigma}_{x_1|y_1}^{(j_1 k_1)} = \boldsymbol{\Sigma}_{x_1}^{(j_1 k_1)} - \boldsymbol{\Sigma}_{x_1 y_1}^{(j_1 k_1)} (\boldsymbol{\Sigma}_{y_1}^{(j_1 k_1)})^{-1} \boldsymbol{\Sigma}_{y_1 x_1}^{(j_1 k_1)}$$

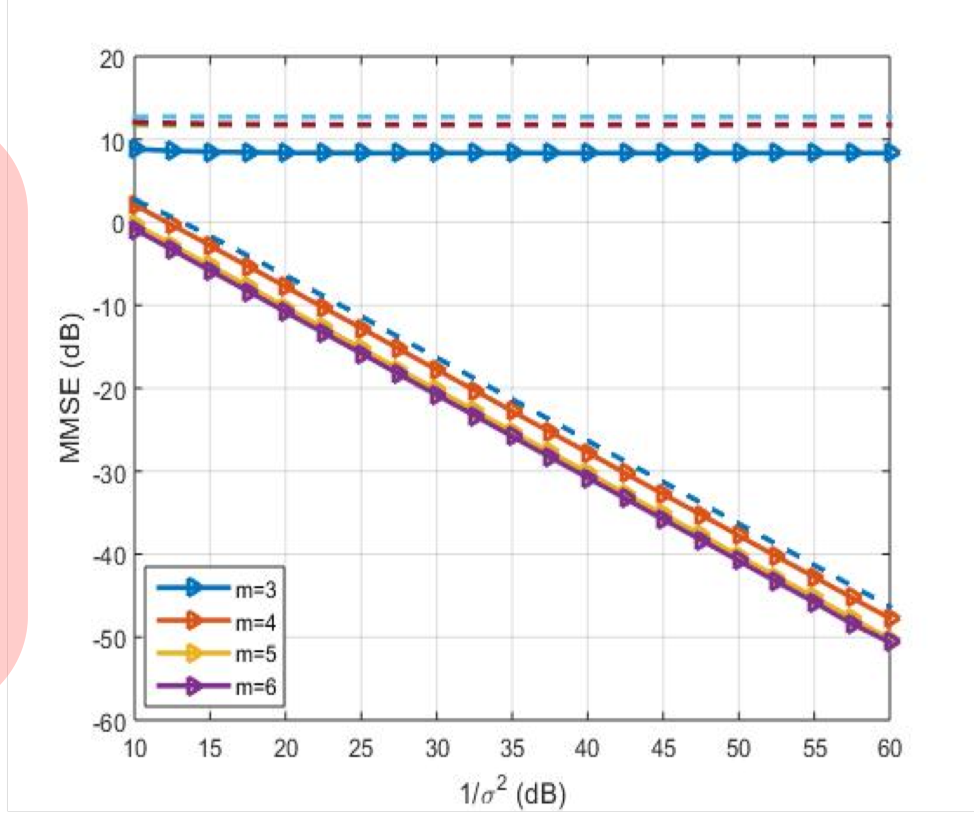
$$\boldsymbol{\Sigma}_{x_2|y_2}^{(j_2 k_2)} = \boldsymbol{\Sigma}_{x_2}^{(j_2 k_2)} - \boldsymbol{\Sigma}_{x_2 y_2}^{(j_2 k_2)} (\boldsymbol{\Sigma}_{y_2}^{(j_2 k_2)})^{-1} \boldsymbol{\Sigma}_{y_2 x_2}^{(j_2 k_2)}$$

The sufficient condition is only one measurement away from the necessary condition.

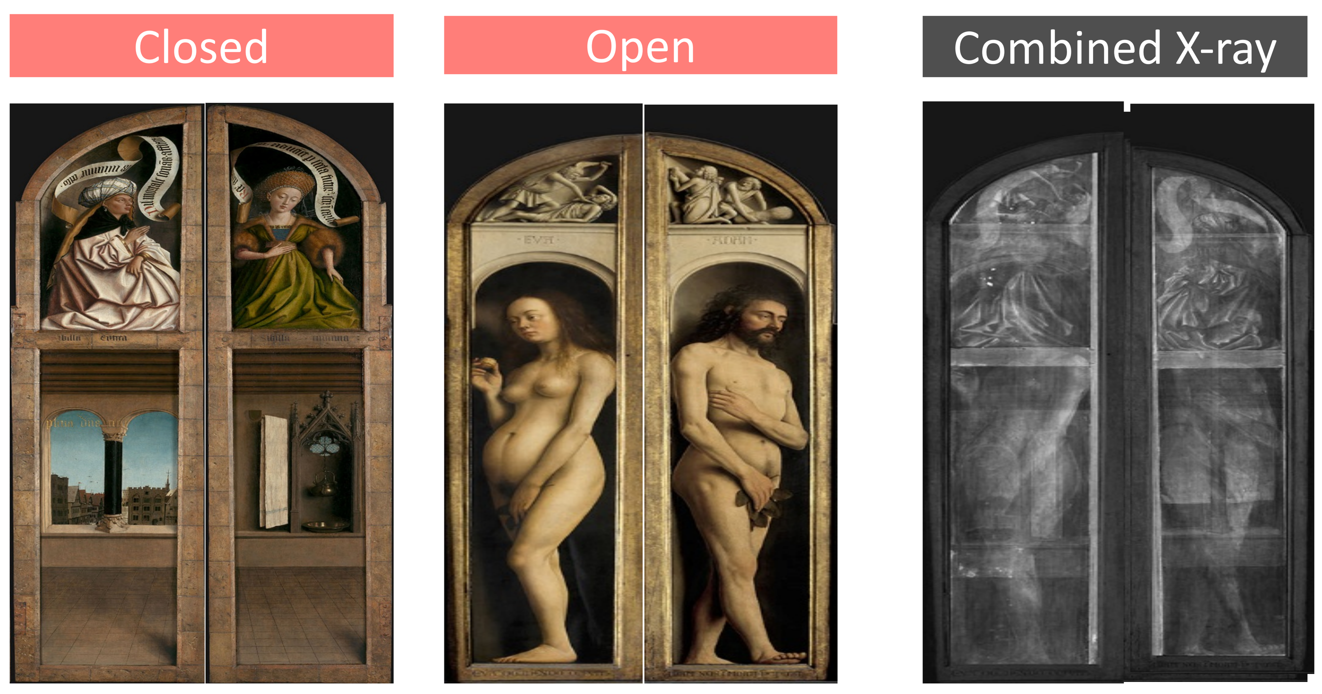
Numerical Results

- **Synthetic Data**

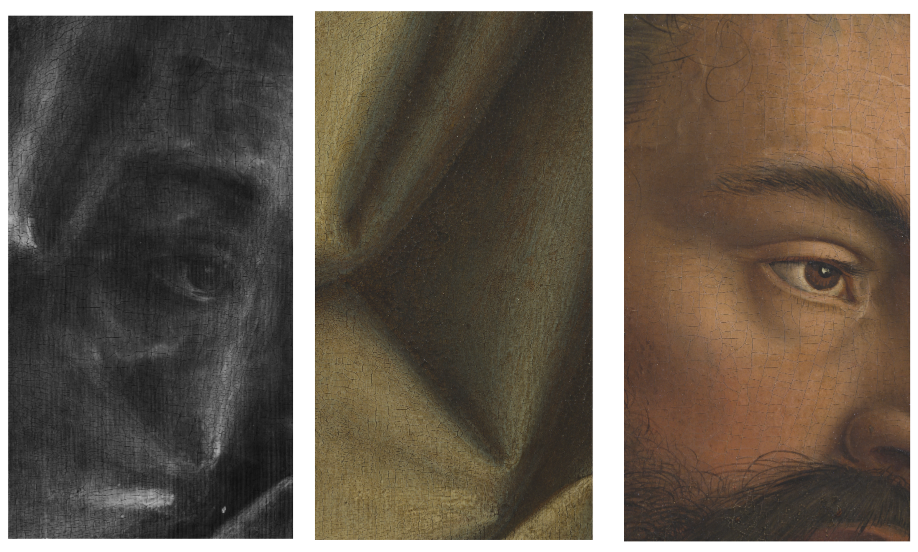
- Setup
  $n_x = n_{y_1} = n_{y_2} = 10$, $|C_1| = |C_2| = |S_1| = |S_2| = 1$, all means are zero and all covariance matrices are generated randomly such that $r_{x_1 y_1} = r_{x_2 y_2} = 5$ and $r_{y_1} = r_{y_2} = 3$.



- **Real Data (Ghent Altarpiece Dataset)**



- Given $\mathbf{w} = \boldsymbol{\Phi}(\boldsymbol{x_1} + \boldsymbol{x_2})$ (Combined X-rays from Ghent Altarpiece)
- Recover $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ (individual X-rays)
- In the presence of side informations $\boldsymbol{y_1}$ and $\boldsymbol{y_2}$ (visible Images)



References

[1] N. Deligiannis, J. F. C. Mota, B. Cornelis, M. R. D. Rodrigues, and I. Daubechies, "Multi-modal dictionary learning for image separation with application in art investigation," *IEEE Transactions on Image Processing*, vol. 26, no. 2, pp. 751–764, Feb 2017.

[2] F. Renna, L. Wang, X. Yuan, J. Yang, G. Reeves, R. Calderbank, L. Carin, and M. R. D. Rodrigues, "Classification and reconstruction of high-dimensional signals from low-dimensional features in the presence of side information," *IEEE Transactions on Information Theory*, vol. 62, no. 11, pp. 6459–6492, Nov 2016.

[3] F. Renna, R. Calderbank, L. Carin, and M. R. D. Rodrigues, "Reconstruction of signals drawn from a gaussian mixture via noisy compressive measurements," *IEEE Transactions on Signal Processing*, vol. 62, no. 9, pp. 2265–2277, May 2014