Generalized Approximate Message Passing for Unlimited Sampling of Sparse Signals

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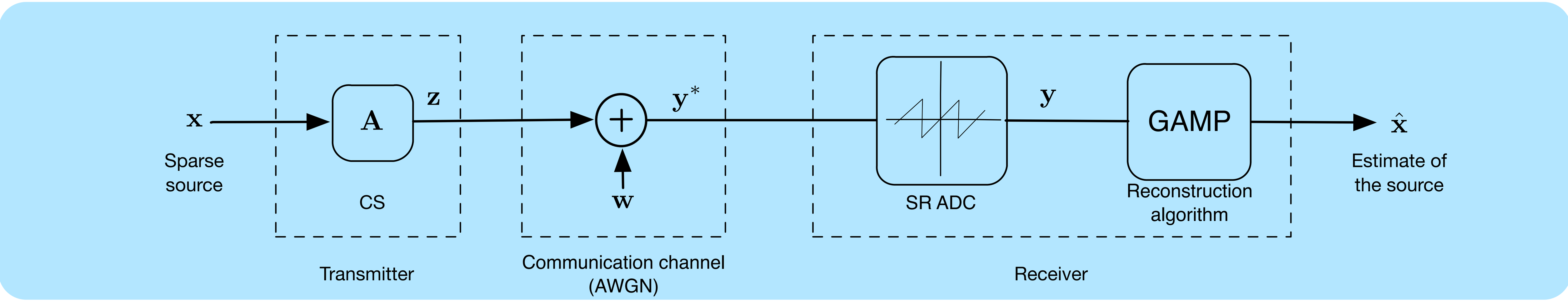


Introduction

Compressed Sensing

- In Compressed Sensing (CS) we take M linear measurements $\{y_i\}_{i=1}^{M}$ of an N-dimensional K-sparse (has at most K nonzero components) vector $\mathbf{x}$, according to

$$\mathbf{y} = \mathbf{A}\mathbf{x} \ (+\mathbf{w})$$

- Recovery is possible if $\mathbf{A} \in \mathbb{R}^{M\times N}$ satisfies the Restricted Isometry Property (RIP)
- Matrices whose elements are randomly drawn from a (sub-)Gaussian distribution satisfy RIP with high probability

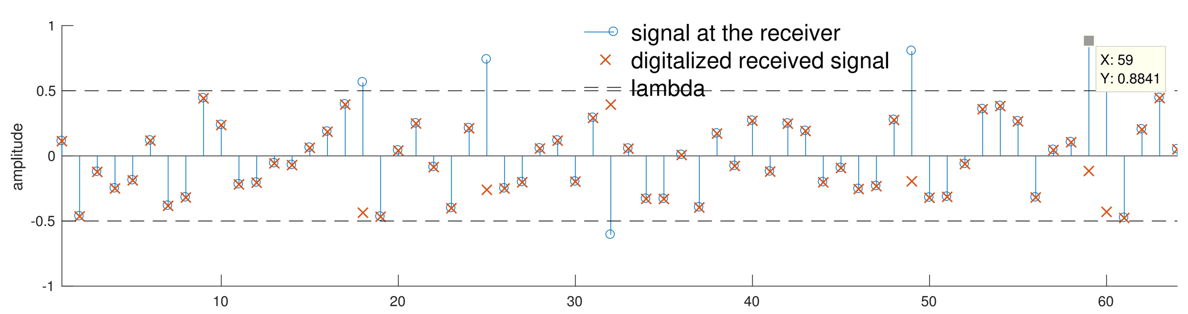
Main Idea and Contributions

- We want to reduce the effects of finite dynamic range
- Image and audio processing, bio-medical applications etc.
- Bhandari et al. propose digitalizing bandlimited signals with a self-reset (SR) analog to digital converter (ADC) defined by

$$\mathcal{M}_\lambda(t) = 2\lambda\left(\left[\!\left[\frac{t}{2\lambda} + \frac{1}{2}\right]\!\right] - \frac{1}{2}\right)$$

where $[\![t]\!] \triangleq t - \lfloor t \rfloor$ is the remainder of the division t and λ

- Perfect recovery of a bandlimited signal from its discrete samples is possible if the sampling period $T \leq (2\pi e)^{-1}$
- A. Bhandari et al. provide sufficient conditions for perfect recovery of a K-sparse signal from its low-pass filtered version
- We take CS measurements and digitalize them with a SR ADC
- The main contributions:
  - ▶ Consider a new way of digitalizing CS measurements
  - ▶ Apply the known GAMP framework
  - ▶ Provide closed-form expressions for the nonlinear steps



Unlimited sampling of AWGN corrupted CS measurements

Bernoulli-Gaussian Mixture Prior

We assume an i.i.d. source vector where each component x_j of $\mathbf{x}$ is a realization of a Bernoulli-Gaussian distributed random variable

$$p_{x_j}(x_j) = (1-\epsilon)\delta(x_j) + \epsilon\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{1}{2\sigma^2}x_j^2},$$

with ϵ being the probability of a non-zero value and σ^2 being the variance of the zero-mean Gaussian distribution

Measurement Model

- We obtain M noisy CS measurement according to

$$\mathbf{y} = \mathcal{M}_\lambda(\mathbf{A}\mathbf{x} + \mathbf{w}).$$

- We assume i.i.d. noise vector $\mathbf{w}$ where each $w_i \sim \mathcal{N}(0, \sigma_w^2)$

GAMP for Unlimited sampling

Why GAMP?

- GAMP is very appealing for its efficiency and accurate recovery
- It approximates the computationally intractable high-dimensional integration involved with calculating

$$\hat{\mathbf{x}} \approx \mathbb{E}\{\mathbf{x} \mid \mathbf{y}\}$$

- It allows to model the quantization as a probabilistic channel with unquantized input and quantized output
- It allows to incorporate measurement noise in the model

The Steps of GAMP

- At $t = 0$, the algorithm is initialized according to (the far right values correspond to the Bernoulli-Gaussian mixture prior)

$$\hat{\mathbf{x}}^0 = \mathbb{E}\{\mathbf{x}\} = 0, \quad \mathbf{v}_x^0 = \mathrm{var}\{\mathbf{x}\} = (1-\gamma)\sigma^2, \quad \hat{\mathbf{s}}^0 = 0_{M\times 1}$$

- At every iteration $t = 1, 2, \ldots$ compute the measurement and estimation updates

$$\mathbf{v}_p^t = (\mathbf{A}\bullet\mathbf{A})\mathbf{v}_x^{t-1} \qquad \mathbf{v}_r^t = \left((\mathbf{A}\bullet\mathbf{A})^T\mathbf{v}_s^t\right)^{-1}$$
$$\hat{\mathbf{p}}^t = \mathbf{A}\hat{\mathbf{x}}^{t-1} - \mathbf{v}_p^t \bullet \hat{\mathbf{s}}^{t-1} \qquad \hat{\mathbf{r}}^t = \hat{\mathbf{x}}^{t-1} + \mathbf{v}_r^t \bullet (\mathbf{A}^T\hat{\mathbf{s}}^t)$$

$$\hat{\mathbf{s}}^t = \mathrm{F}_1(\mathbf{y}, \hat{\mathbf{p}}^t, \mathbf{v}_p^t) \qquad \hat{\mathbf{x}}^t = \mathrm{G}_1(\hat{\mathbf{r}}^t, \mathbf{v}_r^t; p_x)$$
$$\mathbf{v}_s^t = \mathrm{F}_2(\mathbf{y}, \hat{\mathbf{p}}^t, \mathbf{v}_p^t) \qquad \mathbf{v}_x^t = \mathrm{G}_2(\hat{\mathbf{r}}^t, \mathbf{v}_r^t; p_x)$$

- The functions $\mathrm{F}_1(\cdot)$, $\mathrm{F}_2(\cdot)$, $\mathrm{G}_1(\cdot)$ and $\mathrm{G}_2(\cdot)$ are applied component-wise and are given by

$$\mathrm{F}_1(y, \hat{p}, v_p) = \frac{\mathbb{E}\{z \mid y\} - \hat{p}}{v_p} \qquad \mathrm{G}_1(\hat{r}, v_r; p_x) = \mathbb{E}\{x \mid \hat{r}\}$$
$$\mathrm{F}_2(y, \hat{p}, v_p) = \frac{v_p - \mathrm{var}\{z \mid y\}}{v_p^2} \qquad \mathrm{G}_2(\hat{r}, v_r; p_x) = \mathrm{var}\{x \mid \hat{r}\}$$

- The first and the second moment of $z|y$ and $x|\hat{r}$ are evaluated with respect to

$$p_{z|y} \propto p_{y|z}(\cdot \mid \cdot)p_z(\cdot) \quad \text{and} \quad p_{x|\hat{r}} \propto g(\cdot; \hat{r}, v_r)p_x(\cdot)$$

where $z \sim \mathcal{N}(\hat{p}, v_p)$

- Stop iterating if $\|\hat{\mathbf{x}}^t - \hat{\mathbf{x}}^{t-1}\|_2 < \varepsilon\,\|\hat{\mathbf{x}}^t\|_2$ with a small $\varepsilon > 0$ (e.g. $\varepsilon = 10^{-2}$) or when $t \geq t_{\max}$
- GAMP nonlinear operators are computed for the specific input and output channels and given in the paper

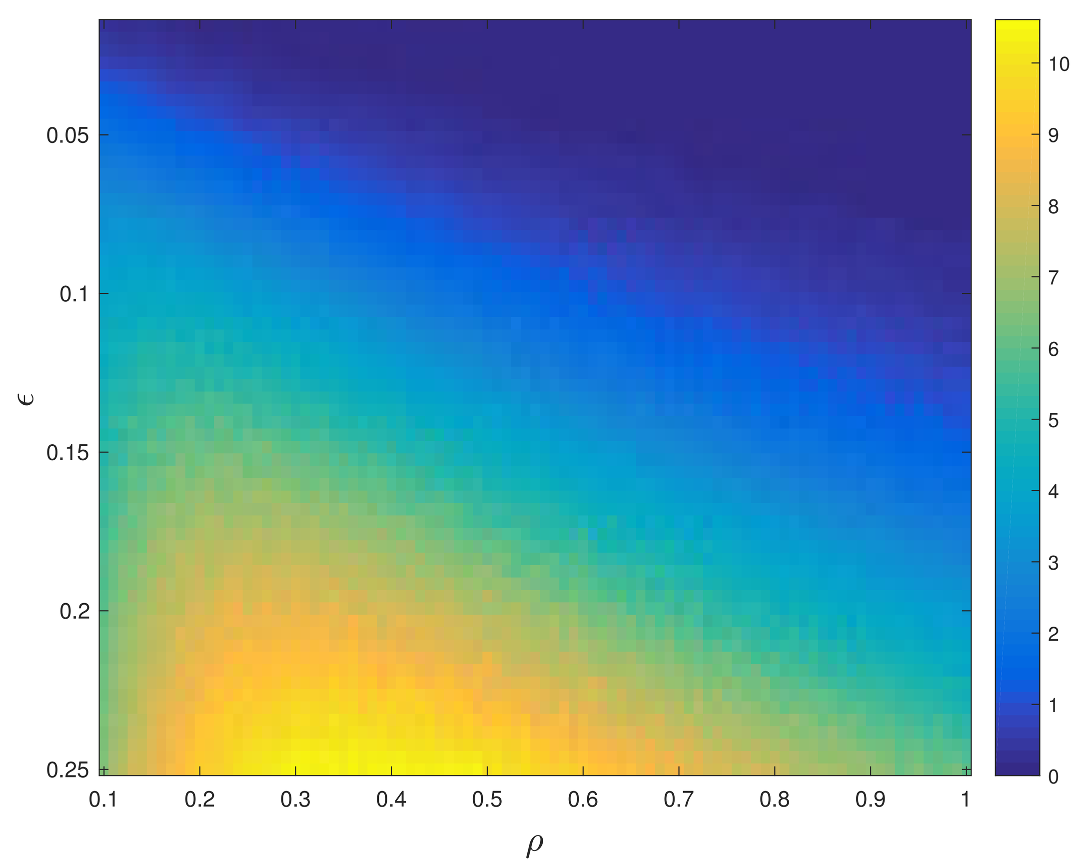
Numerical Results

Simulation Setup

- We averaged our results over 4000 independent realizations of the source vector $\mathbf{x}$, the sensing matrix $\mathbf{A}$ and the AWGN $\mathbf{w}$
- In each simulation we fix $N = 256$, and acquire $n = \rho N$ measurements of the $K = \epsilon N$ sparse vector
- Each CS measurement vector is corrupted with AWGN noise with power $\sigma_w^2 = 10^{-\mathrm{SNR}/10}$, where the SNR is defined as

$$\mathrm{SNR/dB} = 10\log_{10}\{\|\mathbf{y}^*\|^2/\|\mathbf{w}\|^2\}$$

- In the noiseless case $\mathrm{SNR} = \infty$. The SR ADC threshold $\lambda = 1$
- Successful recovery := $\mathrm{MSE} \leq -30\mathrm{dB}$



Results

- Noiseless Case: There's a clear phase transition (PT) between unsuccessful (blue) and successful (yellow) regions
- Classical CS algorithms completely fail when $\|\epsilon_g\|_0 \neq 0$
- GAMP can cope with distortion; the PT is almost linear in ϵ
- Noisy Case: The PT curve shifted to the right lower corner
- More measurements are needed when the CS measurements are corrupted with AWGN ($\mathrm{SNR} = 20\mathrm{dB}$) before digitalization

Conclusion

- For certain choice of the signal parameters, the GAMP is able to successfully recover a sparse signal from folded measurements
- Unlike the classical algorithms for recovery of sparse signals from folded measurements, the GAMP algorithm can cope with the noise introduced by a communication channel