Time Delay Estimation: Applications and Algorithms

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Introduction

What is Time Delay Estimation?

Time delay estimation (TDE) refers to finding the time-differences-of-arrival between signals received at an array of sensors.

A general signal model is:

$$r_i[n] = \alpha_i s[n - \tau_i] + q_i[n], \quad i = 1, 2 \cdots, M, \; n = 0, 1, \cdots, N-1$$

where $r_i[n]$ is the received signal, $s[n]$ is the the signal-of-interest with α_i and τ_i being the gain/attenuation and propagation delay, and $q_i[n]$ is the noise, at the ith sensor.

There are M sensors, and at each sensor, N observations are collected.

Given $\{r_i[n]\}$, the task of TDE is to estimate

$$\tau_{i,j} = -\tau_{j,i} = \tau_i - \tau_j, \quad i > j, \ i, j = 1, 2, \cdots, M$$

Although there are $M(M-1)/2$ delays, there are only $M-1$ nonredundant parameters because of $\tau_{i,j} = \tau_{i,k} - \tau_{j,k}$. A nonredundant set can be $\tau_{i,1}$, $i = 2, 3, \cdots, M$.

As an example, we have delay estimates $\hat{\tau}_{2,1}$, $\hat{\tau}_{3,1}$ and $\hat{\tau}_{3,2}$ for $M = 3$, and the accuracy of $\hat{\tau}_{2,1}$ and $\hat{\tau}_{3,1}$ can be improved by making use of their relationship with $\hat{\tau}_{3,2}$ [1].

Similar terminologies include time-difference-of-arrival (TDOA) estimation, time-of-arrival (TOA) estimation, and time-of-flight (TOF) estimation.

Types of Time Delay Estimation

Considering the fundamental issue where $M = 2$, TDE can be classified as active and passive systems [2].

Active TDE means that $s[n]$ is known and the problem can be formulated as follows. Given

$$r[n] = \alpha s[n-D] + q[n], \quad n = 0, 1, \cdots, N-1$$

The task is to find the TDOA D using $s[n]$ and $r[n]$.

In passive TDE, $s[n]$ is unknown, and the signal model is:

$$r_1[n] = s[n] + q_1[n], \quad r_2[n] = \alpha s[n-D] + q_2[n], \quad n = 0, 1, \cdots, N-1$$

That is, we aim to find D using $r_1[n]$ and $r_2[n]$.

Applications

Many science and engineering problems are related to TDE:

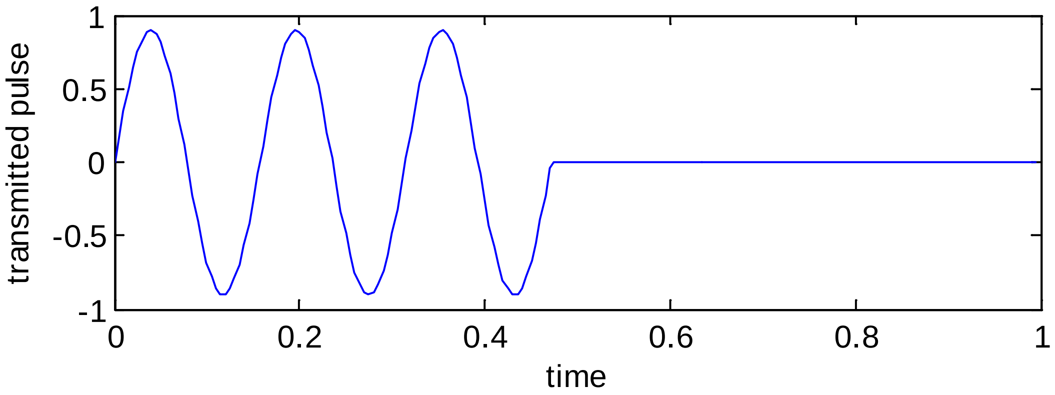
➢ Radar Ranging

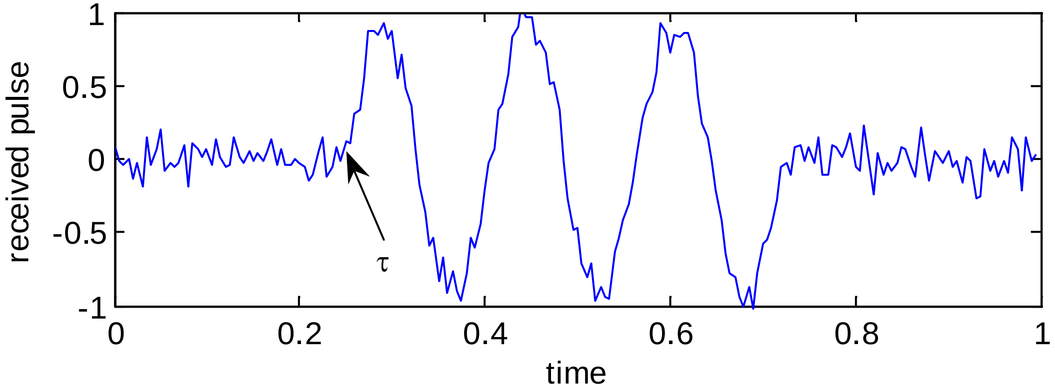
Suppose a radar system transmits an electromagnetic pulse $s(t)$, which is then reflected by an object at a range of R, causing an echo to be received.

The received $r(t)$ is scaled, delayed and noisy version of $s(t)$:

$$r(t) = \alpha s(t-\tau) + w(t)$$

It is clear that the time delay $\tau > 0$ is round trip propagation time.



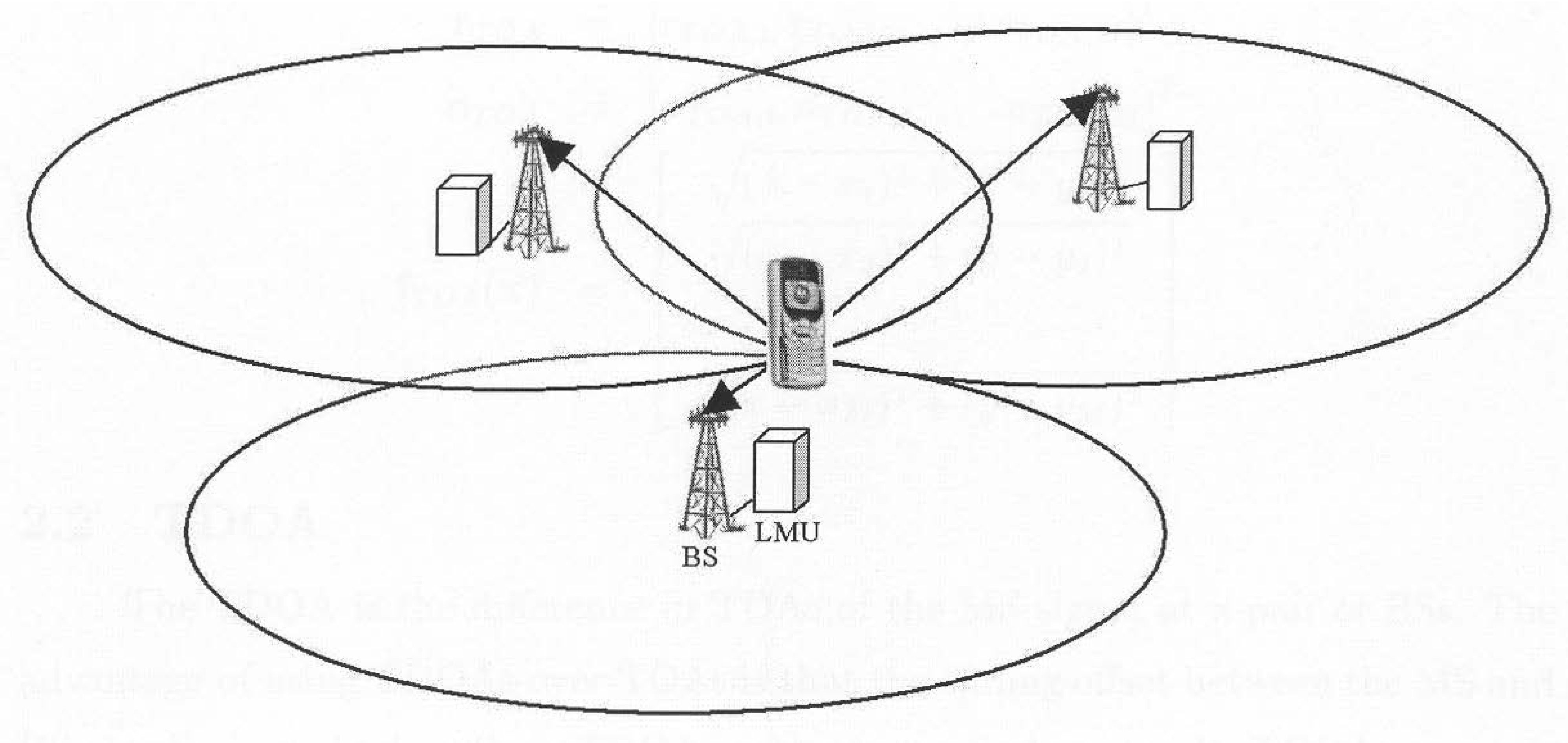


Via estimating τ, R can be obtained using the relationship:

$$\tau \cdot c = 2R$$

where c is the signal propagation speed.

➢ Wireless Location

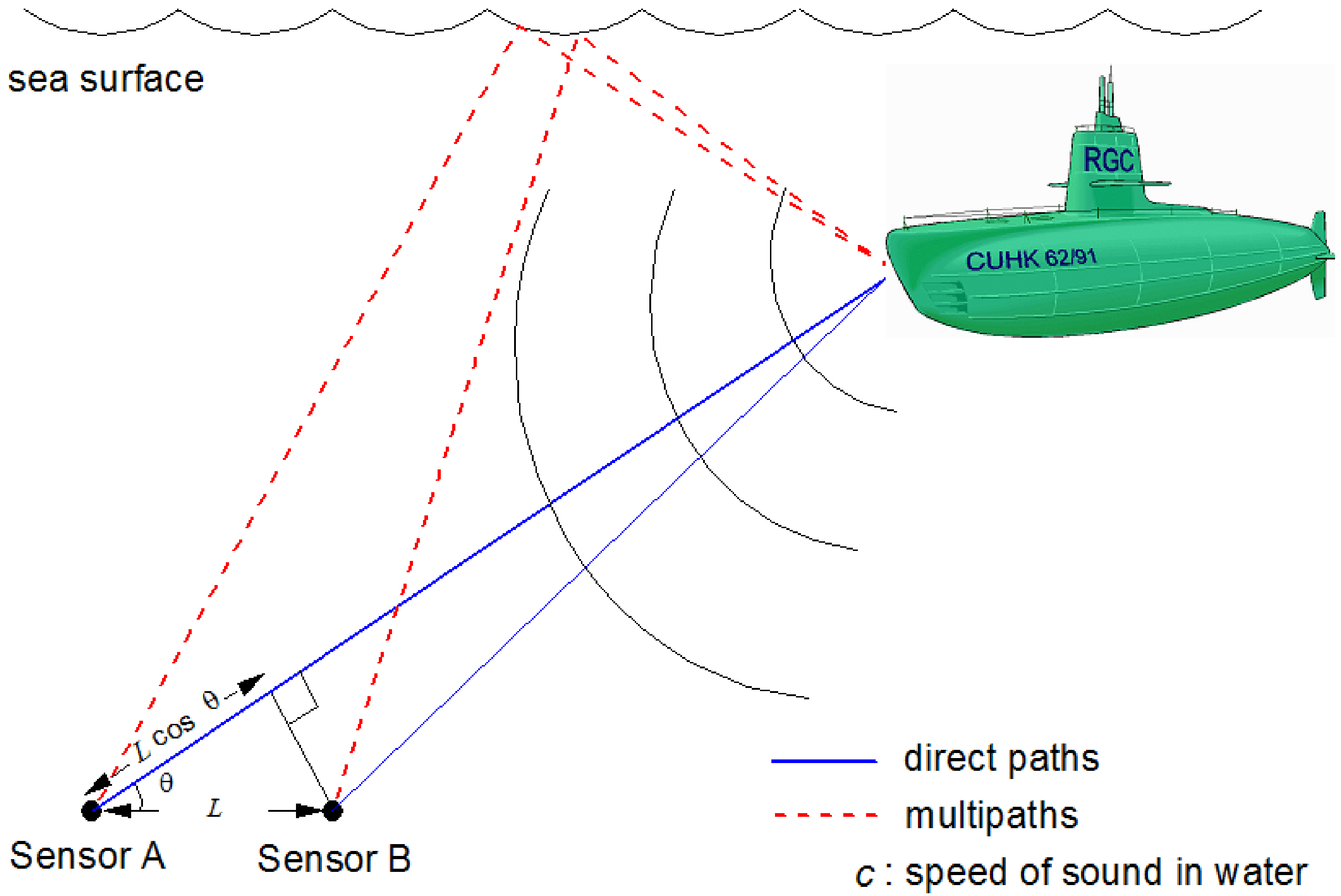


If we know one-way propagation time of the signal traveling between mobile station and base station (BS), then the target position can be obtained using three BSs.

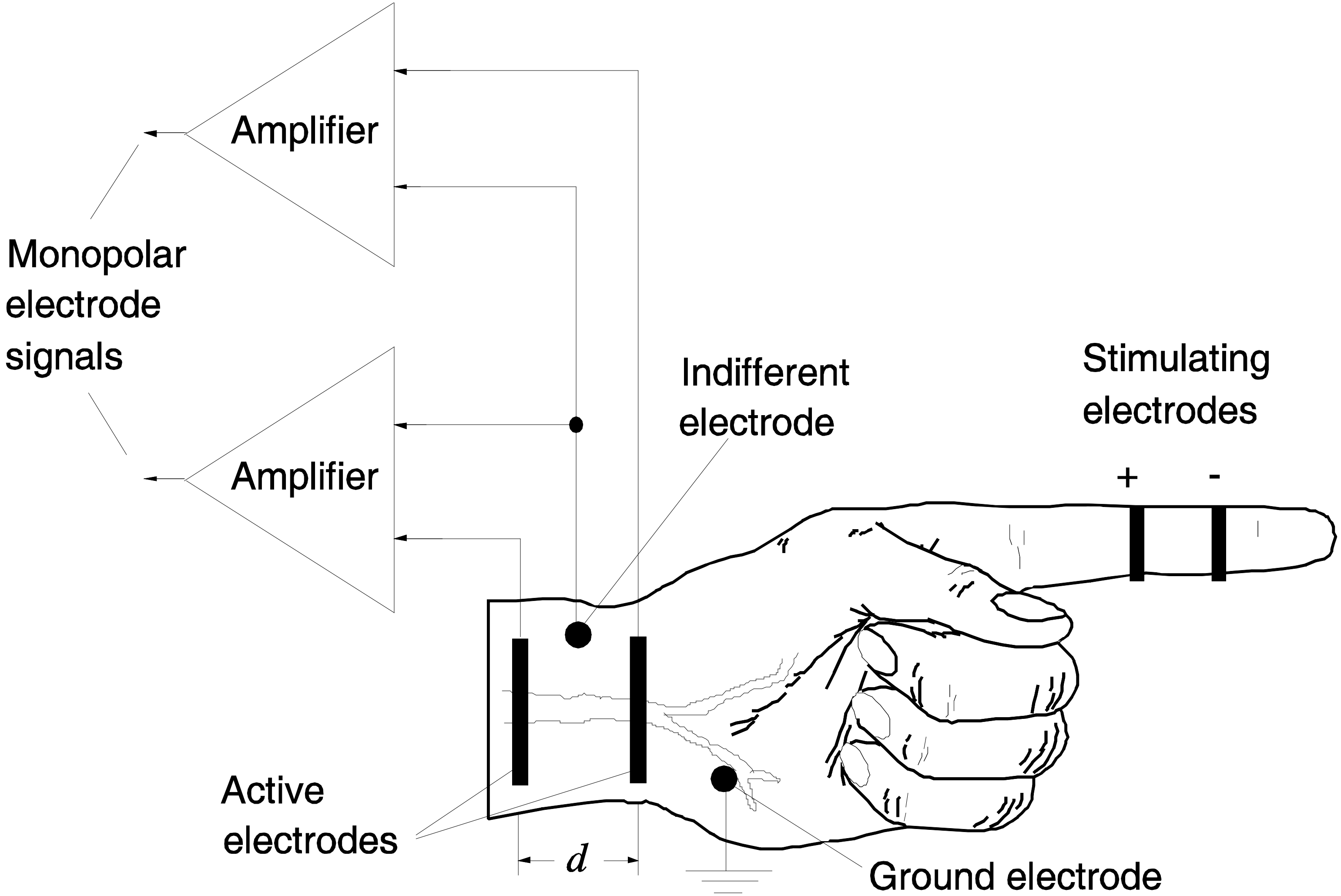
➢ Sonar Direction Finding

For a far-field target, the delay D can be converted to direction-of-arrival:

$$\theta \approx \cos^{-1}\left(\frac{cD}{L}\right)$$



➢ Nerve Conduction Velocity Estimation



The speed is given by the TDOA between the electrode signals divided by d.

- Delay Acquisition for Satellite Navigation
- Particle Size and Speed Estimation in Laser Anemometry
- Digital Pre-distortion for Power Amplifiers
- Beamforming
- Reflection Seismology in Exploration Geophysics
- Speaker Localization
- Control and Synchronization in Chaos Systems
- Sensor Calibration for Augmented Reality

Algorithms for Random Signals

We consider the following model:

$$r_1[n] = s[n] + q_1[n], \quad r_2[n] = \alpha s[n-D] + q_2[n], \qquad n = 0, 1, \cdots, N-1$$

The task is to find D using $r_1[n]$ and $r_2[n]$.

For simplicity, it is assumed that $s[n]$, $q_1[n]$ and $q_2[n]$ are zero-mean white processes with variances σ_s^2 and $\sigma_q^2 = \sigma_{q_1}^2 = \sigma_{q_2}^2$, which are independent of each other.

Starting with the minimum mean square error (MMSE) criterion, the cost function to be minimized is:

$$J_{\mathrm{MMSE}}(\tilde{\alpha}, \tilde{D}) = \mathbb{E}\{\left(r_2[n] - \tilde{\alpha} r_1[n - \tilde{D}]\right)^2\}$$

The MMSE estimates of α and D can be easily computed as:

$$\hat{\alpha} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_q^2}\alpha, \quad \hat{D} = D$$

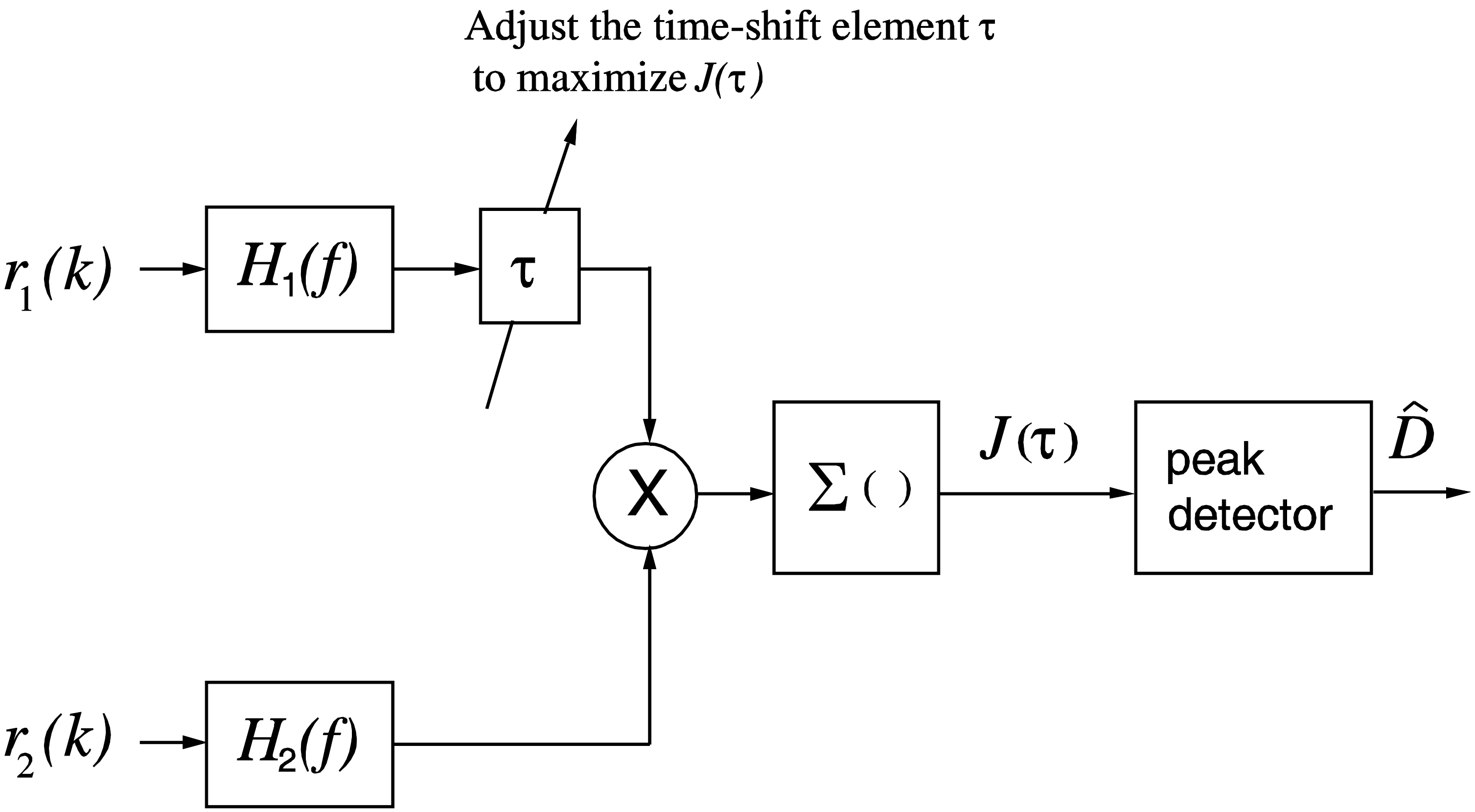
implying that unbiased delay estimation is achieved.

Expanding $J_{\mathrm{MMSE}}(\tilde{\alpha}, \tilde{D})$ and noting that $\mathbb{E}\{r_1^2[n - \tilde{D}]\} = \sigma_s^2 + \sigma_q^2$, it is seen that $\hat{D}$ can also be obtained from:

$$\hat{D} = \arg\max_{\tilde{D}} R_{2,1}(\tilde{D}), \quad R_{2,1}(\tilde{D}) = \mathbb{E}\{r_2[n]r_1[n - \tilde{D}]\}$$

where corresponds to the cross correlation method.

Note that the general form is known as generalized cross correlator [3]-[4] where the prefilters are employed to enhance the frequency bands where the signal is strong and to attenuate the bands where the noise is excessive.



To produce $r_1[n-\tilde{D}]$ from $r_1[n]$, we can apply the interpolation formula [5]:

$$r_1[n-D] = \sum_{i=-\infty}^{\infty} r_1[n-i]\text{sinc}(i-D) \approx \sum_{i=-P}^{P} r_1[n-i]\text{sinc}(i-D)$$

where

$$\text{sinc}(v) = \frac{\sin(\pi v)}{\pi v}$$

Note that $P > |D|$ should be chosen sufficiently large to reduce the delay modeling error [6].

To avoid varying $\tilde{D}$ in the maximization of $R_{2,1}(\tilde{D})$, an alternative is to employ $R_{2,1}(p)$, $p = -P, -P+1, \cdots, P$, which is easily computed.

Using the interpolation formula, we straightforwardly obtain:

$$R_{2,1}(p) = \alpha\sigma_s^2 \text{sinc}(p-D), \quad p = -P, -P+1, \cdots, P$$

In practice, $R_{2,1}(p)$ is replaced by its estimate, $\hat{R}_{2,1}(p)$, which is computed using finite samples of $r_1[n]$ and $r_2[n]$.

The TDOA estimate can be obtained using sinc interpolation [6] of $\{\hat{R}_{2,1}(p)\}$:

$$\hat{D}_{\text{CC,sinc}} = \arg\max_{\hat{D}} \sum_{p=-P}^{P} \hat{R}_{2,1}(p) \text{sinc}(p - \tilde{D})$$

However, $\hat{D}_{\text{CC,sinc}}$ is biased for finite P.

To circumvent the delay bias, we use a least squares (LS) fit to process $\{\hat{R}_{2,1}(p)\}$ via minimizing:

$$J_{\rm LS}(\tilde{\gamma}, \tilde{D}) = \sum_{p=-P}^{P} \left(\hat{R}_{2,1}(p) - \tilde{\gamma}{\rm sinc}(p - \tilde{D}) \right)^2$$

where $\tilde{\gamma}$ is the optimization variable for $\gamma = \alpha\sigma_s^2$.

As γ is easily solved with a closed-form expression, we can remove $\tilde{\gamma}$ in $J_{\rm LS}(\tilde{\gamma}, \tilde{D})$, resulting in the TDOA estimate:

$$\hat{D}_{\rm CC,LS} = \arg\min_{\tilde{D}} \sum_{p=-P}^{P} \left(\hat{R}_{2,1}(p) - \frac{\sum_{i=-P}^{P} \hat{R}_{2,1}(i){\rm sinc}(i - \tilde{D})}{\sum_{i=-P}^{P} {\rm sinc}^2(i - \tilde{D})} {\rm sinc}(p - \tilde{D}) \right)^2$$

As it is not practical to generate a perfect $r_1[n-\tilde{D}]$ in the MMSE criterion, a second methodology is to model $\tilde{\alpha} r_1[n-\tilde{D}]$ using a noncausal FIR filter [5]-[7]:

$$W(z) = \sum_{p=-P}^{P} w_p z^{-p}$$

Employing the interpolation formula and white property of $s[n]$, $q_1[n]$ and $q_2[n]$, the MMSE solution is:

$$w_p = \arg\min_{\tilde{w}_p} \mathbb{E}\left\{\left(r_2[n] - \sum_{p=-P}^{P} \tilde{w}_p r_1[n-p]\right)^2\right\} = \frac{\sigma_s^2}{\sigma_s^2+\sigma_q^2}\alpha \mathrm{sinc}(i-D)$$

which aligns with the estimate using $r_1[n-\tilde{D}]$ where an infinite filter length is required.

Sample correlations are used in practice, which results in:

$$\begin{bmatrix} \hat{w}_{-P} \\ \hat{w}_{-P+1} \\ \vdots \\ \hat{w}_P \end{bmatrix} = \begin{bmatrix} \hat{R}_{1,1}(0) & \hat{R}_{1,1}(1) & \cdots & \hat{R}_{1,1}(2P) \\ \hat{R}_{1,1}(1) & \hat{R}_{1,1}(0) & \cdots & \hat{R}_{1,1}(2P-1) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{R}_{1,1}(2P) & \hat{R}_{1,1}(2P-1) & \cdots & \hat{R}_{1,1}(0) \end{bmatrix}^{-1} \begin{bmatrix} \hat{R}_{1,2}(-P) \\ \hat{R}_{1,2}(-P+1) \\ \vdots \\ \hat{R}_{1,2}(P) \end{bmatrix}$$

Employing sinc interpolation, we have [6]:

$$\hat{D}_{\mathrm{FIR,sinc}} = \arg\max_{\hat{D}} \sum_{p=-P}^{P} \hat{w}_p \mathrm{sinc}(p - \tilde{D})$$

Employing LS regression, we have [7]:

$$\hat{D}_{\mathrm{FIR,LS}} = \arg\min_{\tilde{D}} \sum_{p=-P}^{P} \left(\hat{w}_p - \frac{\sum_{i=-P}^{P} \hat{w}_i \mathrm{sinc}(i - \tilde{D})}{\sum_{i=-P}^{P} \mathrm{sinc}^2(i - \tilde{D})} \mathrm{sinc}(p - \tilde{D}) \right)^2$$

Consider unconstrained optimization problem via minimizing a differentiable cost function, that is:

$$\hat{x} = \arg\min_{\tilde{x}} J(\tilde{x})$$

Based on Taylor's series expansion of $J(\hat{x})$, the mean and mean square error (MSE) of the estimate are derived as [8]:

$$E\{\hat{x}\} \approx x - \frac{E\{J'(x)\}}{E\{J''(x)\}}$$

$$\mathrm{MSE}(\hat{x}) = E\{(\hat{x} - x)^2\} \approx \frac{E\{(J'(x))^2\}}{(E\{J''(x)\})^2}$$

Applying the formulae, we have shown [7]:

$$\hat{D}_{\rm FIR,LS} \approx D$$

and

$$\mathrm{MSE}(\hat{D}_{\rm FIR,LS}) \approx \mathrm{var}(\hat{D}_{\rm FIR,LS}) \approx \frac{1+2\mathrm{SNR}}{N\mathrm{SNR}^2 \displaystyle\sum_{j=-P}^{P} \mathrm{sinc}'^2(j-D)}$$

where $\mathrm{SNR} = \sigma_s^2/\sigma_q^2$.

Note that

$$\sum_{j=-\infty}^{\infty} \mathrm{sinc}'^2(j-D) = \frac{\pi^2}{3}$$

Hence for sufficiently large P, we have:

$$\mathrm{var}(\hat{D}_{\mathrm{FIR,LS}}) \approx \frac{3(1+2\mathrm{SNR})}{\pi^2 N \mathrm{SNR}^2}$$

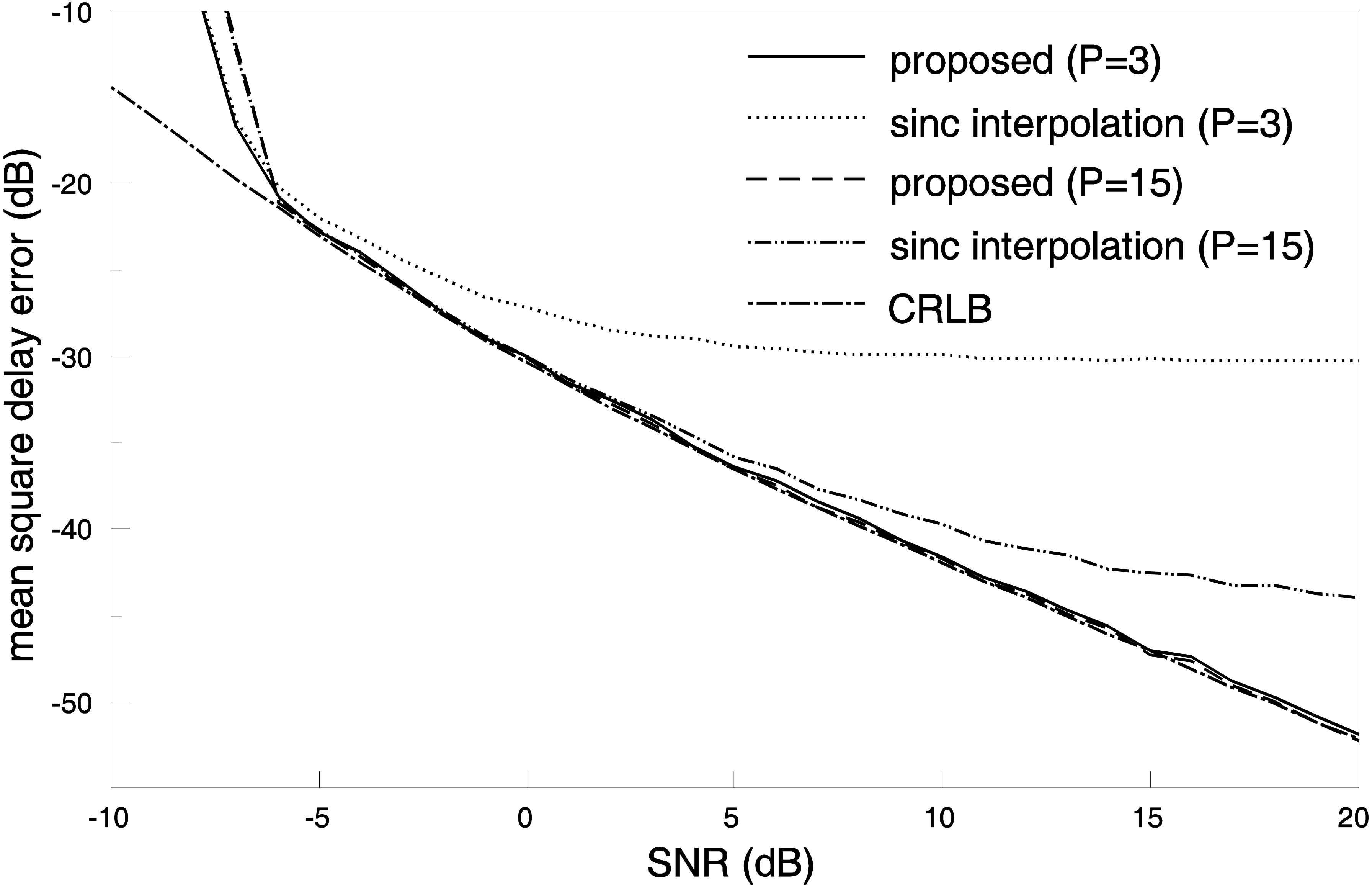
which is equal to Cramer-Rao lower bound (CRLB) [2],[4] for Gaussian data.

When P is chosen large enough, it can also be shown that:

$$\hat{D}_{\mathrm{FIR,sinc}} \approx \hat{D}_{\mathrm{CC,sinc}} \approx \hat{D}_{\mathrm{CC,LS}} \approx D$$

$$\mathrm{var}(\hat{D}_{\mathrm{FIR,sinc}}) \approx \mathrm{var}(\hat{D}_{\mathrm{CC,sinc}}) \approx \mathrm{var}(\hat{D}_{\mathrm{CC,LS}}) \approx \frac{3(1+2\mathrm{SNR})}{\pi^2 N \mathrm{SNR}^2}$$

That is, all of them are asymptotically optimum.



Comparison between $\hat{D}_{\mathrm{FIR,sinc}}$ and $\hat{D}_{\mathrm{FIR,LS}}$ [7]

Adaptive realizations of these batch mode techniques can be designed accordingly. For example, based on the MMSE solution of w_p, the corresponding least mean squares (LMS) algorithm can be [9]:

$$
\begin{aligned}
\hat{\alpha}(k+1) &= \hat{\alpha}(k) - \frac{\mu_\alpha}{2}\frac{\partial e^2[k]}{\partial \hat{\alpha}(k)} \\
&= \hat{\alpha}(k) + \mu_\alpha e[k] \sum_{i=-P}^{P} \operatorname{sinc}(i - \hat{D}(k)) r_1[k-i]
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}(k+1) &= \hat{D}(k) - \frac{\mu_D}{2\hat{\alpha}(k)}\frac{\partial e^2[k]}{\partial \hat{D}(k)} \\
&= \hat{D}(k) - \mu_D e[k] \sum_{i=-P}^{P} \operatorname{sinc}'(i - \hat{D}(k)) r_1[k-i]
\end{aligned}
$$

where

$$e[k] = r_2[k] - \hat{\alpha}(k) \sum_{i=-P}^{P} \text{sinc}(i - \hat{D}(k)) r_1[k-i]$$

Its extension to handle multipath propagation can be found in [10]-[12].

Other LMS algorithms based on FIR filtering modeling or cross correlation include [13]-[15].

Apart from the LMS approach which minimizes the instantaneous squared error $e^2[k]$, we can use recursive least squares (RLS), which minimizes a weighted sum of $e^2[k]$ [16]:

$$\sum_{k=0}^{n} \lambda^{n-k} e^2[k], \quad 0 < \lambda < 1$$

Algorithms for Deterministic Signals

Considering sinusoidal signal, the data model is:

$$r_1[n] = s[n] + q_1[n], \quad r_2[n] = s[n-D] + q_2[n], \qquad n = 0, 1, \cdots, N-1$$

where

$$s[n] = Ae^{j(\omega_0 n + \phi)}, \quad A > 0, \omega_0 \in (-\pi, \pi), \phi \in [0, 2\pi)$$

and now $q_1[n]$ and $q_2[n]$ are zero-mean white complex processes with $\sigma_q^2 = \sigma_{q_1}^2 = \sigma_{q_2}^2$.

The discrete-time Fourier transform (DTFT) can be utilized to estimate D as follows [17].

$$
\begin{aligned}
R_1(e^{j\omega}) &= \sum_{n=0}^{N-1} r_1[n] e^{-j\omega n} \\
&= A e^{j(\phi + (\omega_0 - \omega)(N-1)/2)} \frac{\sin(\frac{(\omega_0-\omega)N}{2})}{\sin(\frac{\omega_0-\omega}{2})} + \sum_{n=0}^{N-1} q_1[n] e^{-j\omega n}
\end{aligned}
$$

At $\omega = \omega_0$:

$$
R_1(e^{j\omega_0}) = NAe^{j\phi}\left[1 + X(e^{j\omega_0})\right]
$$

where

$$
X(e^{j\omega_0}) = \frac{1}{NA} \sum_{n=0}^{N-1} q_1[n] e^{-j(\omega_0 n + \phi)}
$$

For sufficiently high SNR, we obtain:

$$
R_1(e^{j\omega_0}) \approx NAe^{j\phi} \cdot e^{j\Im\{X(e^{j\omega_0})\}} \Rightarrow \angle\{R_1(e^{j\omega_0})\} \approx \phi + \Im\{X(e^{j\omega_0})\}
$$

In a similar manner:

$$\angle\{R_2(e^{\omega_0})\} \approx \phi - \omega_0 D + \Im\{Y(e^{j\omega_0})\}$$

where

$$Y(e^{j\omega_0}) = \frac{1}{NA}\sum_{n=0}^{N-1} q_2[n] e^{-j(\omega_0(n-D)+\phi)}$$

Hence the time delay estimate can be computed as:

$$\hat{D} = \frac{\angle\{R_1(e^{j\omega_0})R_2^*(e^{j\omega_0})\}}{\omega_0}$$

The variance of $\hat{D}$ is derived as:

$$\mathrm{var}(\hat{D}) = \min\left\{\frac{\pi^2}{3\omega_0^2}, \frac{\sigma_q^2}{\omega_0^2 N^2 A^2}\right\}$$

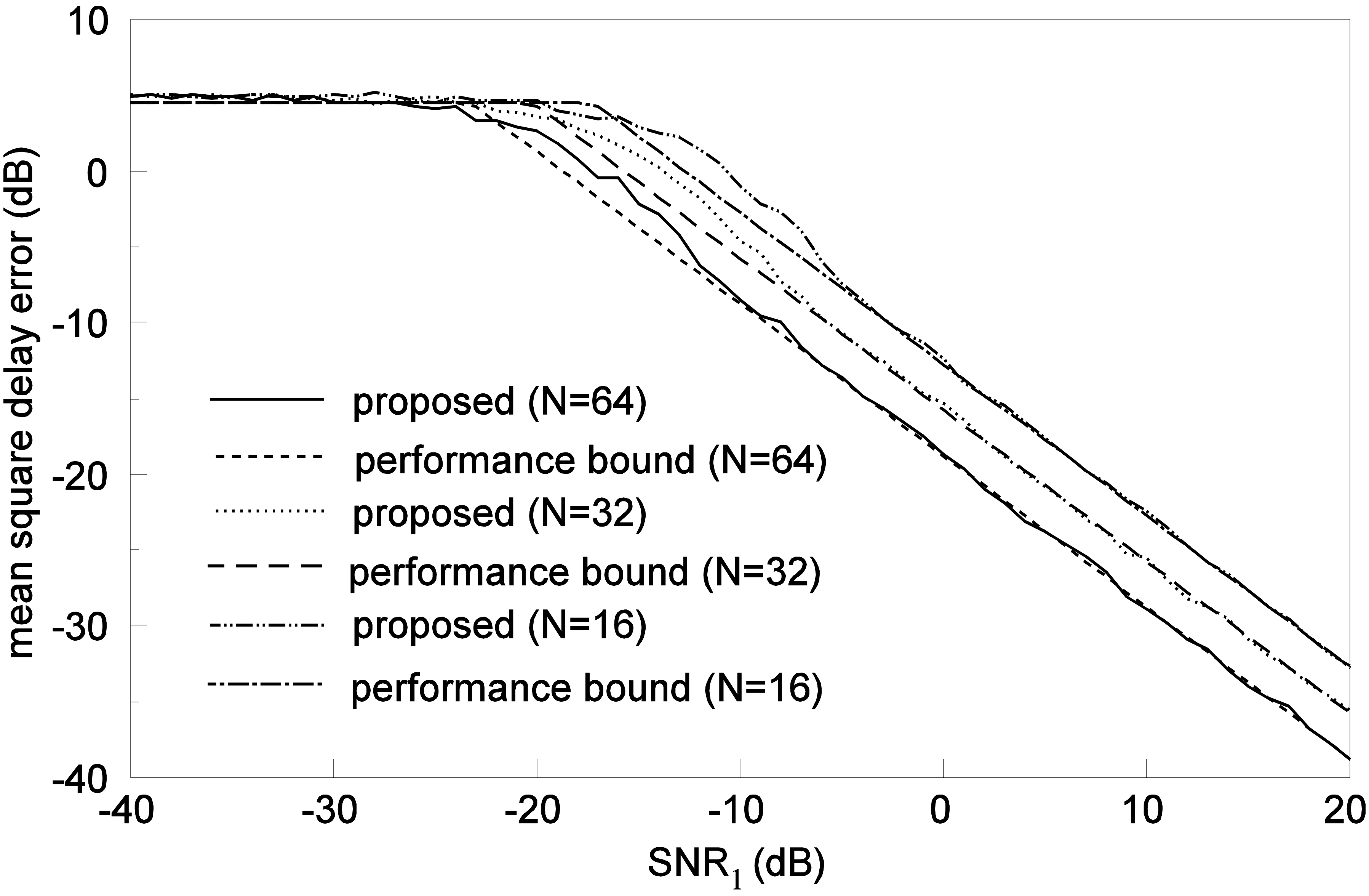
Note that the first term is computed using the knowledge of $\omega_0 D \in (-\pi, \pi)$ while the second term is the CRLB.

This approach has been extended to the case when ω_0 is unknown via locating the peaks of $|R_1(e^{j\omega})|$ and $|R_2(e^{j\omega})|$ and also for real-valued sinusoid [17]-[18].

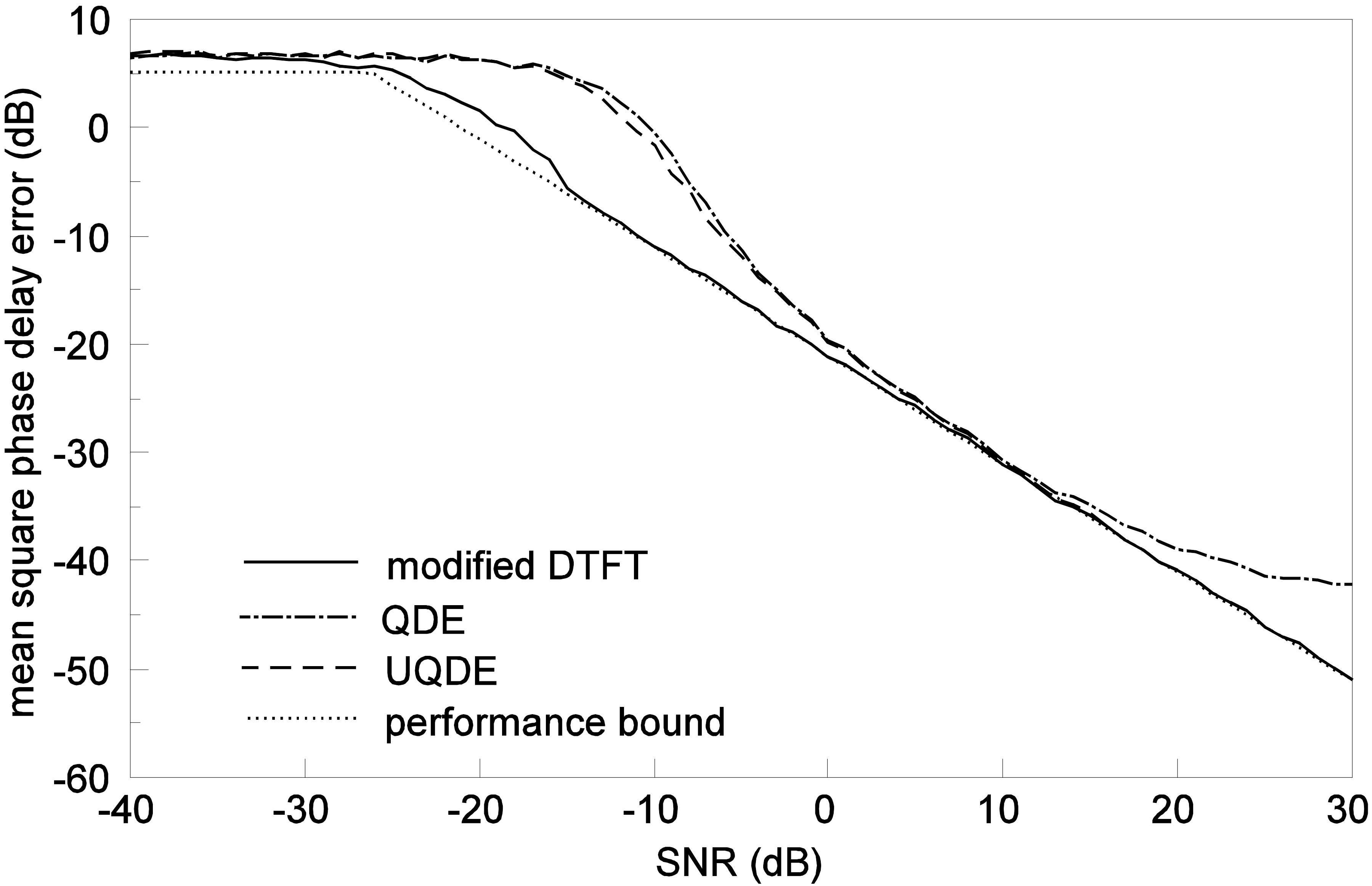
The transform based methodology can also be applied for other deterministic signals. For example, the fractional Fourer transform [19]-[20] has been studied for TDE using chirp signal:

$$s[n] = Ae^{j(\omega_0 n + \nu n^2 + \phi)}$$

where ν denotes the frequency rate.



Performance of DTFT approach versus SNR [17]



Performance of modified DTFT approach versus SNR [18]

For nonstationary scenarios, D may vary with time and adaptive techniques are needed for its tracking.

When a specific sampling frequency is employed, an adaptive FIR filter can be applied to model the TDOA for a sinusoid $s[n] = \cos(\omega_0 n + \phi)$:

$$s[n-D] = h_0 s[n] + h_1 s[n-1], \quad h_0 = \cos(\Omega D), h_1 = \sin(\Omega D)$$

where Ω is the frequency of the analog counterpart [21] or

$$s[n-D] = \frac{2}{L} \sum_{l=0}^{L-1} h_l s[n-l], \quad h_l = \cos(\omega_0 (l-D))$$

where the FIR filter coefficients are samples of a cosine function [22].

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