ENERGY BLOWUP FOR TRUNCATED STABLE LTI SYSTEMS

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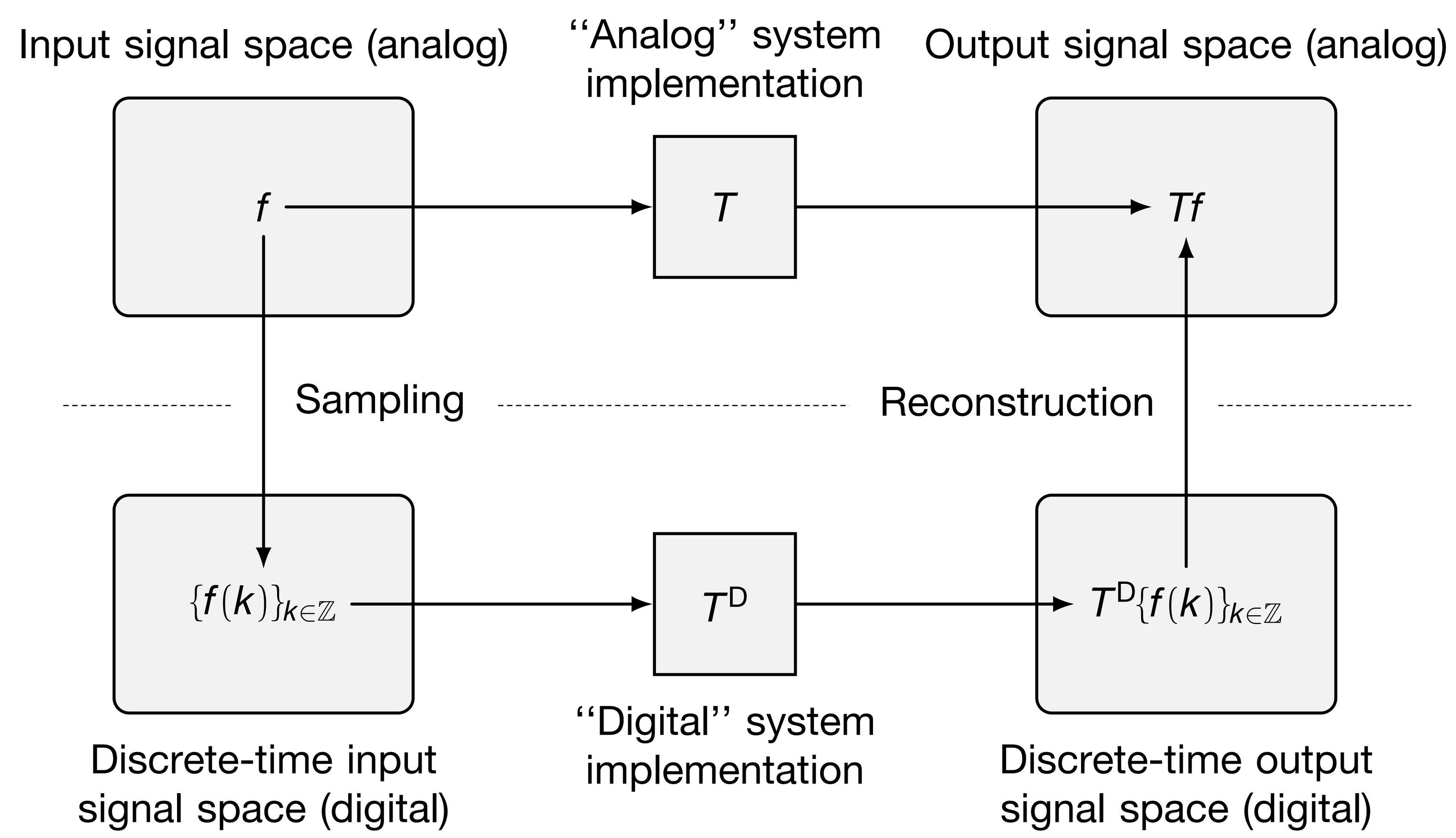
Introduction

Analog vs. digital implementation of a stable LTI system T



Signal reconstruction (classical sampling): Reconstruct a bandlimited signal f from its samples $\{f(k)\}_{k\in\mathbb{Z}}$.

Sampling series: $$f(t) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}$$

System approximation: Approximate the output Tf of a stable LTI system T from the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of the input signal f.

Approximation process 1: $$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k)h_T(t-k)$$

Approximation process 2: $$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(t-k)h_T(k)$$

The interpolation kernel is $h_T = T\,\mathrm{sinc}$.

Notation

Paley–Wiener space $\mathcal{PW}^2_\sigma$: Space of signals f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma}\hat{f}(\omega)\,e^{iz\omega}\,d\omega$, $z\in\mathbb{C}$, for some $\hat{f}\in L^2[-\sigma,\sigma]$. Norm: $\|f\|_{\mathcal{PW}^2_\sigma} = (1/(2\pi)\int_{-\sigma}^{\sigma}|g(\omega)|^2\,d\omega)^{1/2}$.

Stable LTI systems: A linear system $T:\mathcal{PW}^2_\pi\to\mathcal{PW}^2_\pi$ is called stable linear time invariant (LTI) system if:

- T is bounded, i.e., $\|T\| = \sup_{\|f\|_{\mathcal{PW}^2_\pi}\le 1}\|Tf\|_{\mathcal{PW}^2_\pi} < \infty$ and
- T is time invariant, i.e., $\big(Tf(\cdot-a)\big)(t) = (Tf)(t-a)$ for all $f\in\mathcal{PW}^2_\pi$ and $t,a\in\mathbb{R}$.

Representation: For every stable LTI system $T:\mathcal{PW}^2_\pi\to\mathcal{PW}^2_\pi$ there is exactly one function $\hat{h}_T\in L^\infty[-\pi,\pi]$ such that $(Tf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{h}_T(\omega)\hat{f}(\omega)\,e^{i\omega t}\,d\omega$ for all $f\in\mathcal{PW}^2_\pi$. We have $h_T = T\,\mathrm{sinc}$.

Signal Reconstruction

Shannon sampling series:

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)},\quad t\in\mathbb{R}.$$

Convergence in the $\mathcal{PW}^2_\pi$-norm:

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}|f(t)-(S_N f)(t)|^2\,dt = 0$$

Basics of System Approximation

System approximation process 1:
The time variable $t\in\mathbb{R}$ is in the argument of h_T.

$$(T_N^{(1)}f)(t) = \sum_{k=-N}^{N} f(k)h_T(t-k),\quad t\in\mathbb{R}$$

For all $f\in\mathcal{PW}^2_\pi$ and all stable LTI systems $T:\mathcal{PW}^2_\pi\to\mathcal{PW}^2_\pi$ we have:

norm convergence: $$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|(Tf)(t)-(T_N^{(1)}f)(t)\right|^2\,dt = 0$$

and

uniform convergence: $$\lim_{N\to\infty}\max_{t\in\mathbb{R}}\left|(Tf)(t)-(T_N^{(1)}f)(t)\right| = 0.$$

→ The system approximation process $T_N^{(1)}f$ converges in the $\mathcal{PW}^2_\pi$-norm and uniformly on the real axis.

System approximation process 2:
The time variable $t\in\mathbb{R}$ is in the argument of f.

$$(T_N^{(2)}f)(t) = \sum_{k=-N}^{N} f(t-k)h_T(k),\quad t\in\mathbb{R}\qquad (*)$$

For all $f\in\mathcal{PW}^2_\pi$ and all stable LTI systems $T:\mathcal{PW}^2_\pi\to\mathcal{PW}^2_\pi$ we have

uniform convergence: $$\lim_{N\to\infty}\max_{t\in\mathbb{R}}\left|(Tf)(t)-(T_N^{(2)}f)(t)\right| = 0.$$

→ $T_N^{(1)}f$ and $T_N^{(2)}f$ have the same global convergence behavior.

Question

Observation:

$$\sum_{k=-\infty}^{\infty} f(k)h_T(t-k)\quad\text{and}\quad\sum_{k=-\infty}^{\infty} f(t-k)h_T(k)$$

are equiconvergent with respect to the maximum norm.

Question:
Do we have equiconvergence also with respect to the $\mathcal{PW}^2_\pi$-norm?

Linear Structures / Spaceability

Linearity is an important property of signal spaces.
Lineability and spaceability are two mathematical concepts to study the existence of linear structures in general sets.

Definition: A subset S of a Banach space X is said to be lineable if $S\cup\{0\}$ contains an infinite dimensional subspace.
A subset S of a Banach space X is said to be spaceable if $S\cup\{0\}$ contains a closed infinite dimensional subspace.

Easy to see linear structure for convergence:
- f_1, f_2 such that (*) converges $\Rightarrow$ convergence for f_1+f_2

Difficult to show a linear structure for divergence:
- f_1, f_2 such that (*) diverges $\Rightarrow$ not necessarily divergence for f_1+f_2

Example: $f_1 = u_c + u_d$ and $f_2 = u_c - u_d$, where u_c is any signal with convergent and u_d any signal with divergent approximation process.
→ For $f_1 + f_2 = 2u_c$ we do not have divergence.

→ The sum of two signals, each of which leads to divergence, does not necessarily lead to divergence.

Energy Blowup

Let $\mathcal{T}$ denote the set of all stable LTI systems $T:\mathcal{PW}^2_\pi\to\mathcal{PW}^2_\pi$.

Theorem: *There exist an infinite dimensional closed subspace $D_{sig}\subset\mathcal{PW}^2_\pi$ and an infinite dimensional closed subspace $D_{sys}\subset\mathcal{T}$ such that for all $f\in D_{sig}$, $f\not\equiv 0$, and all $T\in D_{sys}$, $T\not\equiv 0$, we have*

$$\limsup_{N\to\infty}\int_{-\infty}^{\infty}\left|\sum_{k=-N}^{N} f(t-k)h_T(k)\right|^2\,dt = \infty.$$

→ The system approximation process $T_N^{(2)}f$ can be divergent with respect to the L^2-norm.
→ The sets of functions in $\mathcal{PW}^2_\pi$ and energetically stable LTI systems having this property are jointly spaceable.

Discussion

For $\mathcal{PW}^1_\pi$ and LTI systems $T:\mathcal{PW}^1_\pi\to\mathcal{PW}^1_\pi$ joint spaceability can be shown even for pointwise divergence. In [BM16] it was sown that for arbitrary $t\in\mathbb{R}$ the sets of functions $f\in\mathcal{PW}^1_\pi$ and stable LTI systems $T:\mathcal{PW}^1_\pi\to\mathcal{PW}^1_\pi$ that satisfy

$$\limsup_{N\to\infty}\left|\sum_{k=-N}^{N} f(t-k)h_T(k)\right| = \infty$$

are jointly spaceable.

[BM16] H. Boche and U. J. Mönich, "Signal and system spaces with non-convergent sampling representation," in *Proceedings of European Signal Processing Conference (EUSIPCO)*, Aug. 2016, pp. 2131–2135