# Source Localization: Applications and Algorithms

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# Introduction

## What is Source Localization?

Determine the position of a source

- Position: $(x, y, z)$, $(x, y)$, latitude and longitude (e.g., HK latitude and longitude are around 22°18' N and 114°10' E,), a point on a map, street number, building, etc.
- Source: target of interest, e.g., mobile phone, tablet PC, person, car, ship, sensor node

Similar terminologies include wireless location, radiolocation and position location

When the source is moving and we want to find the its trajectory versus time, it may be referred to as target tracking or position tracking

## Elements in Positioning

Consider finding the absolute position of a source in terms of $(x, y)$, we need:

- Signals emitted from the source which contain position information
- Sensors or receivers with known coordinates which collect the signals
- An algorithm to compute the location using the received signals. There are two categories:
  - Directly uses the received measurements to obtain the location, and it is referred to as direct approach [1]
  - First extracts location-bearing information such as time-of-arrival and energy estimates from the signals, then base on them to determine $(x, y)$, which may be referred to as two-stage approach [2]

# Applications

## Emergency Assistance

A person with a mobile phone is in an emergency situation but unable to describe his location, e.g.,

- During a hiking activity, a person gets lost from his team and does not know where he is; He calls 911 and then the police can determine his location
- A person sees a terrible accident and he calls the police; However, he is too afraid and cannot tell his position although he should know it

- U.S. Federal Communications Commission (FCC) has mandated the Enhanced 911 (E911) rules where wireless operators are required to provide position information of wireless 911 callers:

  http://transition.fcc.gov/pshs/services/911-services/enhanced911/Welcome.html

  - Phase I: Require wireless operators to report the telephone number of a wireless 911 caller and the location of the antenna or base station that received the call, upon appropriate request by a local Public Safety Answering Point (PSAP)
  - Phase II: Require wireless carriers to provide more precise location information, within 50 to 300 meters in most cases

## Personal Localization and Tracking

The position of a person is known time-by-time, e.g.,

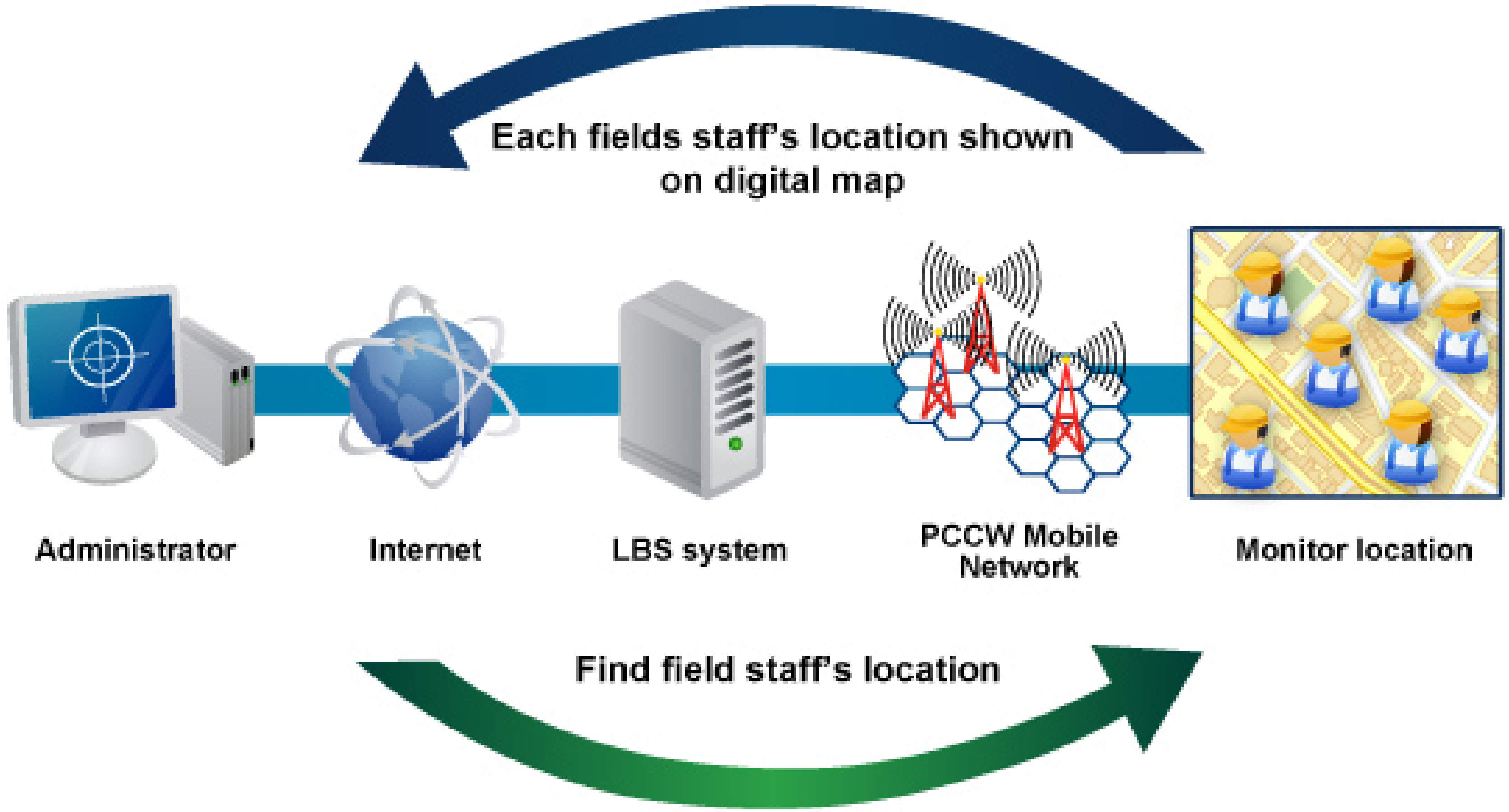
- A parent can know where his child is; For example, the parent is notified when his child has arrived home or school; Or when his child gets lost in a crowded area, his parent can look for him easily http://www.positionlogic.com/industries-gps-tracking-solutions/gps-tracking-solution-child-protection/
- A bodyguard can keep track of an important person to ensure his/her safety, e.g., monitoring the position of a president in a cocktail party
- The position of a mentally impaired person, e.g., elderly with Alzheimer's disease can be known by his relative or caregiver regularly

## Fleet Management and Asset Tracking

A user can track the location and status of a specific group or individual, e.g.,

- Hong Kong Fire Services Department employs the Third Generation Mobilizing System to manage fire engines and ambulances
- Asset tracking represents the supervision of an asset, which includes constant reporting of its position and providing alert to the supervisor and client when the asset leaves a specific area. Company examples include FedEx and UPS
- A manager of a logistic company can know the locations of the company's vehicles and then use the position information to increase the transport efficiency

Hong Kong PCCW provides a service which marks the location of client’s field workforce or packages on a web-based digital map, enabling an easy way to manage resources in terms of people, cargo and vehicles




http://www.pccw-hkt.com/en/Location-Based-Service/

## Travel Services

Provide useful information for tourists, e.g.,

- A tourist can consult a location system where he/she can go shopping or dining nearby, or to find out which is the fastest or cost-effective way to go to his/her next destination. For example, the position or requested route will be shown on the digital map on a mobile phone
- Hong Kong Tourism Board has launched some phone and tablet Apps for visitors with navigation services:

  http://www.discoverhongkong.com/eng/plan-your-trip/travel-kit/mobile-apps.jsp

## Location-based Advertising and Marketing

The idea is to broadcast advertisement and marketing information to users when they enter a geographical area:

- A cinema offers a discount of watching a movie to mobile phone users nearby when the movie will be shown soon
- When a user walks by a store, a special offer advertisement will ring on his mobile device
- Location-based broadcaster sends a mass text to everyone with cell phones in the room at once
  http://www.dailymail.co.uk/sciencetech/article-2653449/The-shocking-car-safety-ad-hijacks-cinemagoers-mobile-phones-exactly-distracting-text-message-be.html

## Location-based Billing

Location-based billing allows a wireless operator to charge different rates to mobile subscribers based on where they are

## Automated Camera Steering

A representative application is video conferencing where we need to automatically capture the face of an active talker via determining his/her position. A microphone array is employed as the receivers for collecting the speech signals

## Distant Speech Acquisition

A distant speech corresponds to low signal-to-noise ratio (SNR) conditions. If we are able to locate its source, beamforming can then be applied to obtain the speech at much higher SNR

## Wireless Sensor Network

A wireless sensor network consists of many small, inexpensive, low-power nodes which collect surrounding data, perform small-scale computations and communicate among their neighbors. It has great potential in numerous remote monitoring and control applications such as:

- Smart home, e.g., when you enter a room, the light will be automatically turned on
- Environmental monitoring, e.g., detection of fire source

# Positioning Principles and Measurement Models

We consider the two-stage approach and assume that the location-bearing information has been extracted from the raw measurements in the first stage. They include:

- Time-of-Arrival (TOA)
- Time-Difference-of-Arrival (TDOA)
- Received Signal Strength (RSS)
- Direction-of-Arrival (DOA)

Given the TOA, TDOA, RSS or DOA estimates, our task is to find source position $(x, y)$

## TOA

One-way signal propagation time between source and receiver

In principle, the signals can be represented as

transmitted signal: $s(t)$

and

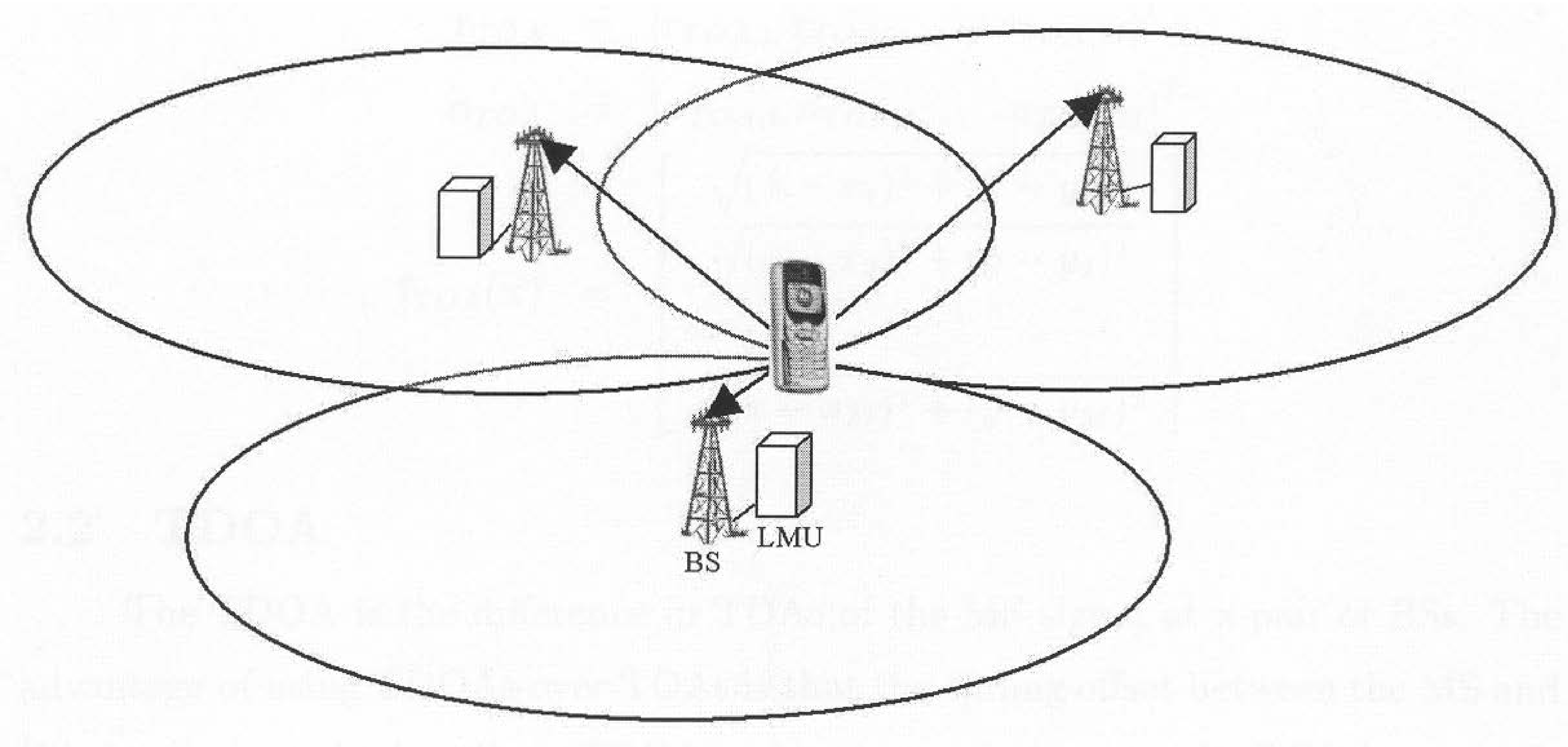
received signal: $s(t-\mathrm{TOA})$

From TOA, the distance between them can be determined:

Distance = $\mathrm{TOA} \times c$

where $c = 3 \times 10^8 \mathrm{ms}^{-1}$ is the speed of light

The target must lie on a circle centered at the receiver with radius $\mathrm{TOA} \times c$




Clock synchronization among source and all receivers is needed

Synchronization in the source is not required if two-way (round-trip) propagation is used

Let $(x_i, y_i)$ be position of the $i$th receiver

Let $t_i$ be the TOA between target and $i$th sensor, we have:

$$t_i = \frac{d_i}{c} \Rightarrow d_i = c \cdot t_i$$

where $d_i$ is the distance between them:

$$d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

Practically, the TOA will be subject to error:

$$r_i = d_i + n_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} + n_i$$

Suppose there are M receivers, we have M equations to solve for the unknown $(x, y)$ with the known $(x_i, y_i)$.

## TDOA

In principle, signals collected at the *m*th and *n*th receivers are

$$m\text{th receiver: } s(t-\mathrm{TOA}_m)$$

and

$$n\text{th receiver: } s(t-\mathrm{TOA}_n)$$

TDOA is the difference in TOAs between two receivers:

$$\mathrm{TDOA}_{m,n} = \mathrm{TOA}_m - \mathrm{TOA}_n$$

Time synchronization is needed among all sensors only

Each TDOA corresponds to one hyperbola and source position is given by intersection of at least two hyperbolae

Let 1st sensor at $(x_1, y_1)$ be the reference and let $t_{i,1}$ be the TDOA of the source between the *i*th and 1st sensors, we have:

$$t_{i,1} = \frac{d_{i,1}}{c} \Rightarrow d_{i,1} = c \cdot t_{i,1}, \qquad\qquad i \neq 1$$

where $d_{i,1}$ is the difference between $d_i$ and $d_1$:

$$d_{i,1} = d_i - d_1 = \sqrt{(x-x_i)^2 + (y-y_i)^2} - \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

Practically, it is not possible to obtain noise-free TDOA:

$$r_{i,1} = d_{i,1} + n_{i,1} = \sqrt{(x-x_i)^2 + (y-y_i)^2} - \sqrt{(x-x_1)^2 + (y-y_1)^2} + n_{i,1}$$

M sensors correspond to (M-1) TDOAs, so we have (M-1) equations to solve for unknown $(x, y)$ with known $(x_i, y_i)$

## RSS

Propagation path loss of the signal traveling from the source to the receiver and their distance can be computed from it.

Let the power transmitted by the source be $P_t$

The received signal power can be expressed as:

$$P_{r,i} \propto \frac{P_t}{\text{distance}^a}$$

where $a \in [2, 6]$ is the propagation constant

Positioning concept is same as TOA but the distance derived from RSS is not very accurate

Clock synchronization is not required

RSS is available at current wireless network, e.g., WiFi (IEEE 802.11) and ZigBee (IEEE 802.15.4)

The received signal power at the *i*th sensor can be modeled as:

$$P_{r,i} = \frac{K_i P_t}{d_i^a} \Rightarrow d_i^a = \frac{K_i P_t}{P_{r,i}}$$

where $P_t$ is the transmitted power, $K_i$ accounts for all factors which affect the received power

However, the noise in RSS is log-normal distributed and the observed RSS in dB is:

$$10\log_{10}(P_{r,i}) = 10\log_{10}(K_i P_t) - 10a\log_{10}(d_i) + w_i$$

or

$$r_i = -a\ln(d_i) + n_i$$

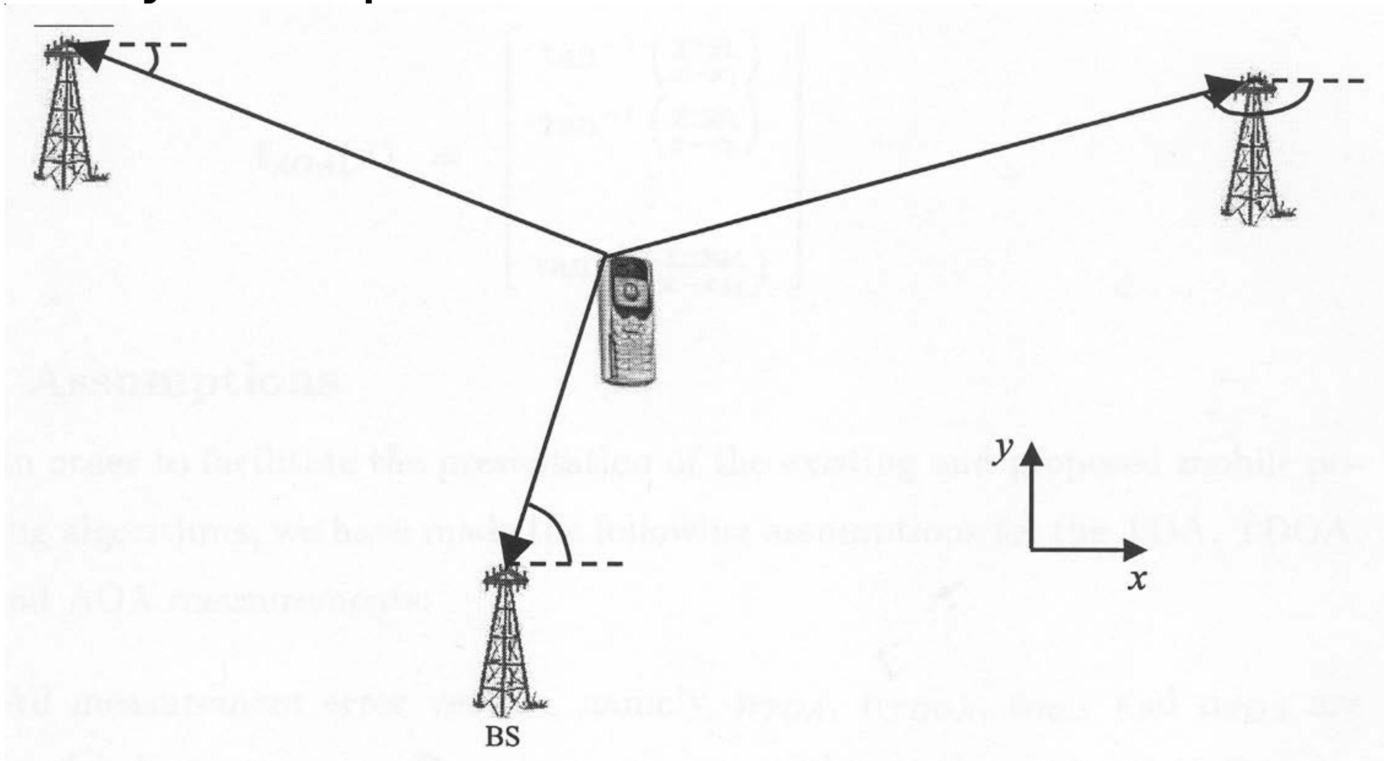
Suppose there are M sensors to measure the RSSs, we have M equations to solve for $(x, y)$ with the known $(x_i, y_i)$

# DOA

Arrival angle of signal from the source at the receiver and each corresponds to a line-of-bearing (LOB)

Intersection of at least two LOBs gives source position

Antenna array is required at the receiver



Let the DOA of signal from the source at the *i*th sensor be $\theta_i$, we get:

$$\tan(\theta_i) = \frac{y - y_i}{x - x_i} \Rightarrow \theta_i = \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right)$$

In the presence of disturbance $n_i$, the noisy DOA measurement is

$$r_i = \theta_i + n_i = \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right) + n_i$$

Suppose there are M receivers to measure the DOAs, we have M equations to solve for $(x, y)$ with the known $(x_i, y_i)$

# Positioning Algorithms

For simplicity, all disturbances $\{n_i\}$ are assumed to be zero-mean and uncorrelated with each other and have the same powers, i.e., independent and identically distributed (i.i.d.)

## Optimum Solution [2]

The basic idea is to estimate the source position via minimizing a nonlinear least squares (NLS) cost function

For TOA equations:

$$r_i = d_i + n_i = \sqrt{(x-x_i)^2 + (y-y_i)^2} + n_i, \qquad i = 1,2,\cdots,M$$

The NLS cost function is

$$J(x,y) = \sum_{i=1}^{M} \left( r_i - \sqrt{(x-x_i)^2 + (y-y_i)^2} \right)^2$$

The position estimate is obtained as:

$$(\hat{x}, \hat{y}) = \arg\min_{x,y} \{J(x, y)\}$$

which means that when $x = \hat{x}$ and $y = \hat{y}$, $J(x, y)$ has the minimum value.

However, finding the minimum is difficult because of nonlinear $J(x, y)$ and thus global solution is not guaranteed

Newton's method can be used with proper initialization:

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{y}_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{x}_k \\ \hat{y}_k \end{bmatrix} - \left. \begin{bmatrix} \frac{\partial^2 J}{\partial x^2} & \frac{\partial^2 J}{\partial x \partial y} \\ \frac{\partial^2 J}{\partial x \partial y} & \frac{\partial^2 J}{\partial y^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial J}{\partial x} \\ \frac{\partial J}{\partial y} \end{bmatrix} \right|_{x=\hat{x}_k,\, y=\hat{y}_k}$$

For TDOA equations:

$$r_{i,1} = \sqrt{(x-x_i)^2 + (y-y_i)^2} - \sqrt{(x-x_1)^2 + (y-y_1)^2} + n_{i,1}, \quad i = 2, \cdots, M$$

The NLS cost function is

$$J(x,y) = \sum_{i=2}^{M} \left( r_i - \sqrt{(x-x_i)^2 + (y-y_i)^2} + \sqrt{(x-x_1)^2 + (y-y_i)^2} \right)^2$$

For RSS equations:

$$r_i = -a \ln\left(\sqrt{(x-x_i)^2 + (y-y_i)^2}\right) + n_i, i = 1, 2, \quad \cdots, M$$

The NLS cost function is

$$J(x,y) = \sum_{i=1}^{M} \left( r_i + a \ln(\sqrt{(x-x_i)^2 + (y-y_i)^2}) \right)^2$$

For DOA equations:

$$r_i = \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right) + n_i, \qquad i = 1,2\cdots, M$$

The NLS cost function is

$$J(x, y) = \sum_{i=1}^{M} \left[ r_i - \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right)\right]^2$$

When all disturbances $\{n_i\}$ are Gaussian distributed, NLS is equivalent to maximum likelihood (ML) approach

NLS is optimum but it is computationally demanding if stochastic approach (e.g., genetic algorithm, particle swarm optimization) or grid search is used and initial estimates are required for gradient-based techniques

# Cramér-Rao Lower Bound (CRLB) [3]

CRLB is performance bound in terms of minimum achievable variance provided by any unbiased estimators

Its derivation requires knowledge of the noise probability density function (PDF) in closed-form

Let $\mathbf{r} = \mathbf{f}(\mathbf{x}) + \mathbf{w}$ be the observations where $\mathbf{f}()$ is a known function, $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_L]^T$ is the vector of parameters of interest, and $\mathbf{w}$ is the noise vector. Denote its PDF by $p(\mathbf{r}; \mathbf{x})$

The CRLB for $\mathbf{x}$ can be obtained in two steps:

- Compute the Fisher information matrix $\mathbf{I}(\mathbf{x})$
- CRLB for $x_l$ is the $(l, l)$ entry of $\mathbf{I}^{-1}(\mathbf{x})$, $l = 1, 2, \cdots, L$

$\mathbf{I}(\mathbf{x})$ has the form of:

$$\mathbf{I}(\mathbf{x}) = \begin{bmatrix} -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial^2 x_1}\right\} & -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial x_1 \partial x_2}\right\} & \cdots & -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial x_1 \partial x_L}\right\} \\ -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial x_2 \partial x_1}\right\} & -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial^2 x_2}\right\} & \cdots & -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial x_2 \partial x_L}\right\} \\ \vdots & \vdots & \vdots & \vdots \\ -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial x_L \partial x_1}\right\} & \cdots & \cdots & -E\left\{\frac{\partial^2 \ln p(\mathbf{r};\mathbf{x})}{\partial^2 x_L}\right\} \end{bmatrix}$$

In our case of two-dimensional positioning, $\mathbf{I}(\mathbf{x})$ is a 2x2 matrix, and we use $[\mathbf{I}^{-1}(\mathbf{x})]_{1,1} + [\mathbf{I}^{-1}(\mathbf{x})]_{2,2}$ as CRLB for location

## Suboptimal but Simple Solutions

1. Linear Least Squares (LLS) [4]-[8]

Key idea is to convert nonlinear equations into linear via introducing extra variable and then apply least squares (LS)

For TOA equations:

$$r_i = d_i + n_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} + n_i, \qquad i = 1,2,\cdots,M$$

Squaring both sides, we have:

$$r_i^{\,2} = (x - x_i)^2 + (y - y_i)^2 + n_i^{\,2} + 2n_i\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

Now the noise component is

$$m_i = n_i^{\,2} + 2n_i\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

Expanding the equation and letting $R = x^2 + y^2$ yields

$$r_i^{\;2} = (x - x_i)^2 + (y - y_i)^2 + m_i$$
$$\Rightarrow r_i^{\;2} = x^2 - 2xx_i + x_i^{\;2} + y^2 - 2yy_i + y_i^{\;2} + m_i$$
$$\Rightarrow -2xx_i - 2yy_i + R = r_i^{\;2} - x_i^{\;2} - y_i^{\;2} - m_i, \quad R = x^2 + y^2$$
$$\Rightarrow -2xx_i - 2yy_i + R \approx r_i^{\;2} - x_i^{\;2} - y_i^{\;2}$$

In matrix form:

$$\mathbf{Ax} \approx \mathbf{b}$$

or

$$\begin{bmatrix} -2x_1 & -2y_1 & 1 \\ \vdots & \vdots & \vdots \\ -2x_M & -2y_M & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ R \end{bmatrix} \approx \begin{bmatrix} r_1^2 - x_1^2 - y_1^2 \\ \vdots \\ r_M^2 - x_M^2 - y_M^2 \end{bmatrix}$$

The LLS cost function is

$$
\begin{aligned}
J(\mathbf{x}) &= (\mathbf{A}\mathbf{x}-\mathbf{b})^T(\mathbf{A}\mathbf{x}-\mathbf{b}) \\
&= \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{A}^T\mathbf{b} - \mathbf{b}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{b} \\
&= \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{2}\mathbf{x}^T\mathbf{A}^T\mathbf{b} - \mathbf{b}^T\mathbf{b}
\end{aligned}
$$

The position estimate is obtained as:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \{J(\mathbf{x})\}$$

which can be easily obtained as

$$
\begin{aligned}
&\frac{dJ(\mathbf{x})}{d\mathbf{x}} = 2\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} - \mathbf{2}\mathbf{A}^T\mathbf{b} = \mathbf{0} \\
&\Rightarrow \mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{b} \\
&\Rightarrow \hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}
\end{aligned}
$$

The solution is simple because

- $\mathbf{A}$ and $\mathbf{b}$ are easy to construct
- Only simple matrix operations are involved

However, it is not optimal because

- The disturbances $\{m_i\}$ are different powers but LLS is optimal only for zero mean i.i.d. noises
- At sufficiently large noise conditions, the means of $\{m_i\}$ are not close to 0 and this also affects LLS accuracy
- The introduced variable $R = x^2 + y^2$ is a function of $x$ and $y$ but this known information is not utilized in LLS

For TDOA equations:

$$r_{i,1} = \sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2} + n_{i,1}, \quad i = 2, \cdots, M$$

$$\Rightarrow r_{i,1} + \sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_i)^2 + (y - y_i)^2} + n_{i,1}$$

Let

$$m_i = n_{i,1}{}^2 + 2n_{i,1}\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

and

$$R_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

Squaring both sides and rearranging the result:

$$(x_i - x_1)(x - x_1) + (y_i - y_1)(y - y_1) + r_{i,1}R_1$$
$$= 0.5\left[(x_i - x_1)^2 + (y_i - y_1)^2 - r_{i,1}{}^2\right] + 0.5m_i$$
$$\approx 0.5\left[(x_i - x_1)^2 + (y_i - y_1)^2 - r_{i,1}{}^2\right], \qquad i = 2,3, \cdots, M$$

In matrix form, we have

$$\mathbf{A}\mathbf{x} \approx \mathbf{b}$$

or

$$\begin{bmatrix} x_2 - x_1 & y_2 - y_1 & r_{2,1} \\ \vdots & \vdots & \vdots \\ x_M - x_1 & y_M - y_1 & r_{M,1} \end{bmatrix} \begin{bmatrix} x - x_1 \\ y - y_1 \\ R_1 \end{bmatrix} \approx \frac{1}{2} \begin{bmatrix} (x_2 - x_1)^2 + (y_2 - y_1)^2 - r_{2,1}^2 \\ \vdots \\ (x_M - x_1)^2 + (y_M - y_1)^2 - r_{M,1}^2 \end{bmatrix}$$

The solution is then:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{b}$$

For RSS equations:

$$r_i = -a\ln(d_i) + n_i, \quad i = 1, 2, \cdots, M$$

Similar to TOA, we can get an estimate of $d_i^2$:

$$\hat{d}_i^2 = \exp\left(-\frac{2r_i}{a} - \frac{2\sigma_i^2}{a^2}\right)$$

such that $E\{\hat{d}_i^2\} = d_i^2$.

In matrix form:

$$\mathbf{Ax} \approx \mathbf{b} \quad \text{or} \begin{bmatrix} -2x_1 & -2y_1 & 1 \\ \vdots & \vdots & \vdots \\ -2x_M & -2y_M & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ R \end{bmatrix} \approx \begin{bmatrix} \hat{d}_1^2 - x_1^2 - y_1^2 \\ \vdots \\ \hat{d}_M^2 - x_M^2 - y_M^2 \end{bmatrix}$$

The solution is:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$

For DOA equations:

$$r_i = \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right) + n_i, \qquad i = 1,2\cdots, M$$

Recall

$$\tan(\theta_i) = \frac{y - y_i}{x - x_i}$$

Taking tangent operation on both sides:

$$r_i - n_i = \tan^{-1}\left(\frac{y - y_i}{x - x_i}\right) \Rightarrow \tan(r_i - n_i) = \frac{y - y_i}{x - x_i} \Rightarrow \tan(r_i) \approx \frac{y - y_i}{x - x_i}$$

Conversion to linear form can be achieved via:

$$\tan(r_i) \approx \frac{y - y_i}{x - x_i} \Rightarrow \frac{\sin(r_i)}{\cos(r_i)} \approx \frac{y - y_i}{x - x_i}$$

$$\Rightarrow \sin(r_i)x - \cos(r_i)y \approx \sin(r_i)x_i - \cos(r_i)y_i, \qquad i = 1,2,\cdots,M$$

In matrix form:

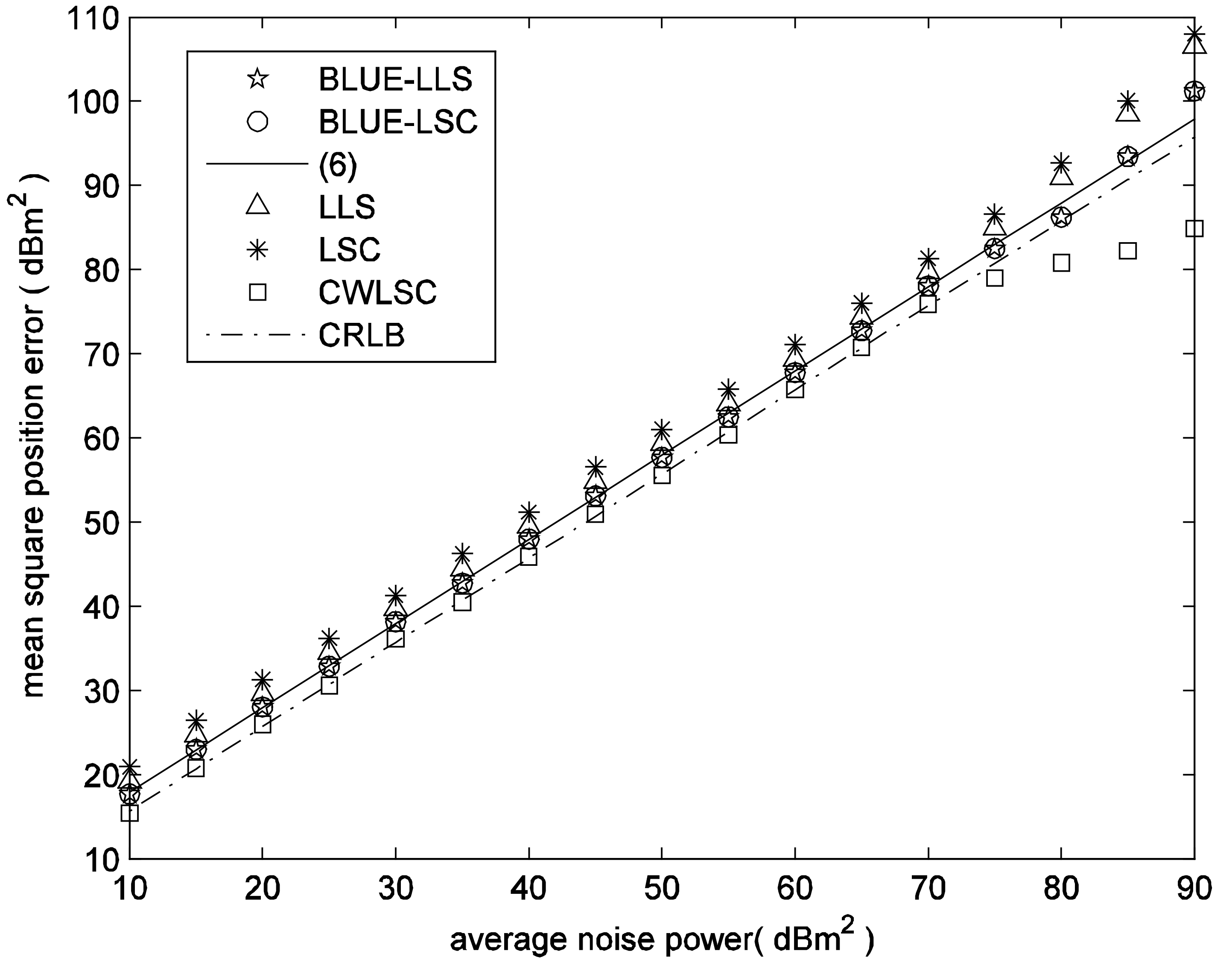
$$\mathbf{Ax} \approx \mathbf{b}$$

or

$$\begin{bmatrix} \sin(r_1) & -\cos(r_1) \\ \vdots & \vdots \\ \sin(r_M) & -\cos(r_M) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \approx \begin{bmatrix} \sin(r_1)x_1 - \cos(r_1)y_1 \\ \vdots \\ \sin(r_M)x_M - \cos(r_M)y_M \end{bmatrix}$$

As a result, the solution is also of the form:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$




Typical mean square error performance of LLS (=LSC)

## 2. Subspace Solution [9]-[12]

Recall TOA model:

$$r_i = d_i + n_i = \sqrt{(x-x_i)^2 + (y-y_i)^2} + n_i, \qquad i = 1,2,\cdots,M$$

Define a rank-2 matrix:

$$\mathbf{D} = \mathbf{X}\mathbf{X}^T$$

where

$$[\mathbf{D}]_{m,n} = 0.5(d_m^2 + d_n^2 - d_{mn}^2)$$

$$d_{mn} = d_{nm} = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}$$

and

$$\mathbf{X} = \begin{bmatrix} x_1 - x & y_1 - y \\ x_2 - x & y_2 - y \\ \vdots & \vdots \\ x_M - x & y_M - y \end{bmatrix}$$

Represent $\mathbf{D}$ using eigenvalue decomposition:

$$\mathbf{D} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T = \mathbf{U_s}\mathbf{\Lambda_s}\mathbf{U_s}^T$$

where

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, 0, \cdots, 0), \;\; \mathbf{\Lambda_s} = \mathrm{diag}(\lambda_1, \lambda_2)$$

$$\mathbf{U} = [\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_M}], \;\; \mathbf{U_s} = [\mathbf{u_1}, \mathbf{u_2}]$$

$\mathbf{X}$ is obtained up to a rotation:

$$\mathbf{X^r} = \mathbf{U}_s\mathbf{\Lambda}_s^{\frac{1}{2}}$$

$\mathbf{X}$ is then determined as:

$$\mathbf{X} = \mathbf{X^r}\mathbf{\Omega} = \mathbf{U}_s\mathbf{\Lambda}_s^{\frac{1}{2}}\mathbf{\Omega}, \qquad \mathbf{\Lambda}_s^{\frac{1}{2}} = \mathrm{diag}(\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}})$$

where

$$\mathbf{\Omega} = (\mathbf{X^r}^T\mathbf{X^r})^{-1}\mathbf{X^r}^T\mathbf{X} = \mathbf{\Lambda}_s^{-\frac{1}{2}}\mathbf{U}_s^T\mathbf{X}$$

$$\Rightarrow \mathbf{X} = \mathbf{U}_s\mathbf{U}_s^T\mathbf{X}$$

In practice, we only have:

$$[\hat{\mathbf{D}}]_{m,n} = 0.5(r_m^2 + r_n^2 - d_{mn}^2)$$

LS estimate of $\mathbf{X}^{\mathrm{r}}$ is:

$$\hat{\mathbf{X}}^{\mathbf{r}} = \arg\min_{\tilde{\mathbf{X}}} ||\hat{\mathbf{D}} - \tilde{\mathbf{X}}\tilde{\mathbf{X}}^T||_F^2 = \mathbf{U}_s\mathbf{\Lambda}_s^{\frac{1}{2}}$$

Hence a signal subspace algorithm is resulted from:

$$\mathbf{X} \approx \mathbf{U}_s\mathbf{U}_s^T\mathbf{X}$$

or

$$\begin{bmatrix} x_1 - x \\ x_2 - x \\ \vdots \\ x_M - x \end{bmatrix} \approx \mathbf{U}_s\mathbf{U}_s^T \begin{bmatrix} x_1 - x \\ x_2 - x \\ \vdots \\ x_M - x \end{bmatrix} \text{ and } \begin{bmatrix} y_1 - y \\ y_2 - y \\ \vdots \\ y_M - y \end{bmatrix} \approx \mathbf{U}_s\mathbf{U}_s^T \begin{bmatrix} y_1 - y \\ y_2 - y \\ \vdots \\ y_M - y \end{bmatrix}$$

is a set of linear equations which can be solved by LLS

Alternatively, using:

$$\mathbf{I} - \mathbf{U}_s\mathbf{U}_s^T = \mathbf{U}_n\mathbf{U}_n^T$$

we have a noise subspace algorithm:

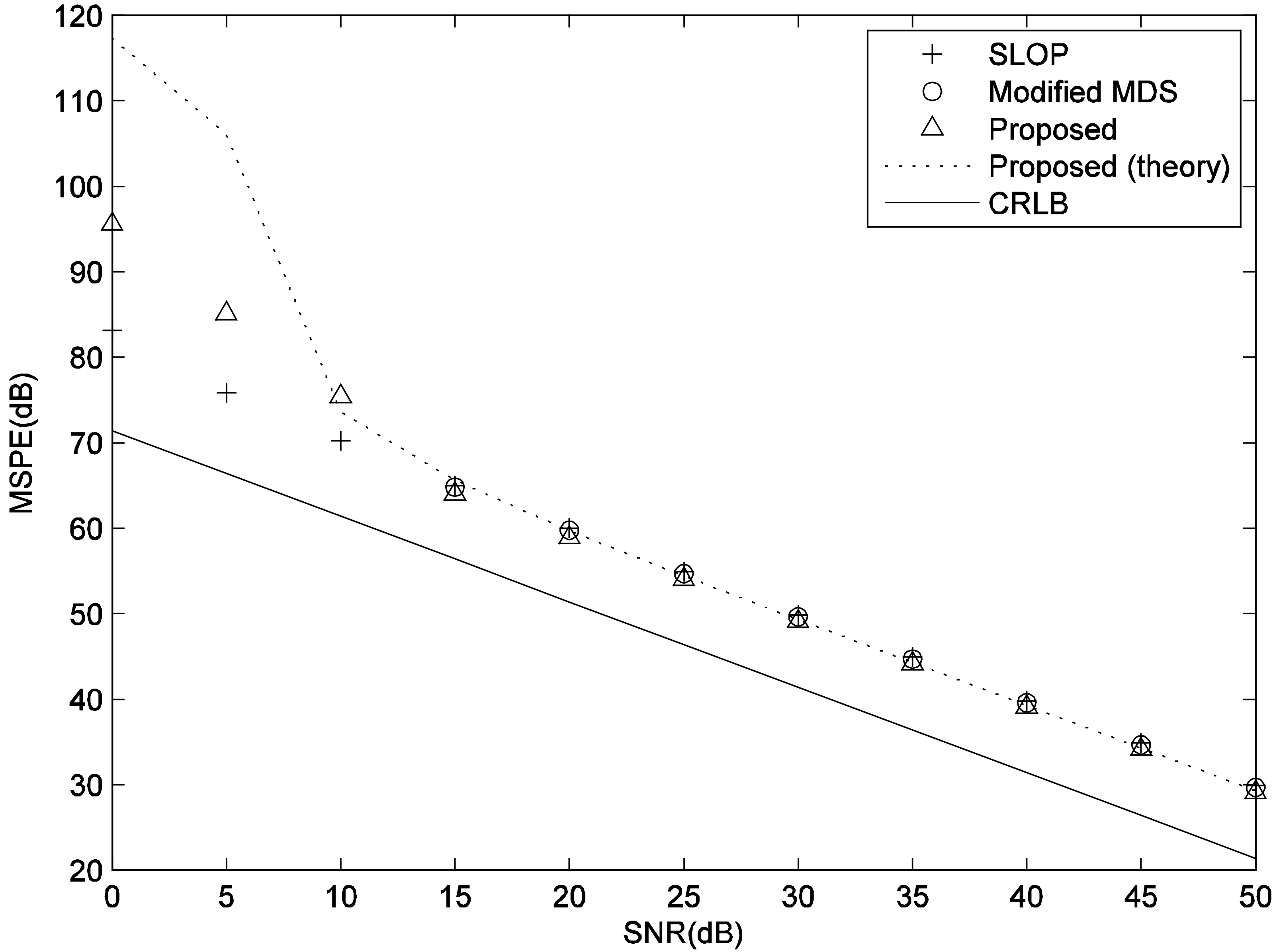
$$\mathbf{U}_n\mathbf{U}_n^T\mathbf{1}[x\ y] \approx \mathbf{U}_n\mathbf{U}_n^T \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix}$$

where

$$\mathbf{U}_n = [\mathbf{u}_3\ \mathbf{u}_4\ \cdots\ \mathbf{u}_M]$$

LS estimate is:

$$[\hat{x}\ \ \hat{y}] = (\mathbf{U}_n\mathbf{U}_n^T\mathbf{1})^\dagger\mathbf{U}_n\mathbf{U}_n^T \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix} = \frac{\mathbf{1}_M^T\mathbf{U}_n\mathbf{U}_n^T}{\mathbf{1}^T\mathbf{U}_n\mathbf{U}_n^T\mathbf{1}} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_M & y_M \end{bmatrix}$$




Typical mean square error performance of subspace method

## Solutions with Improved Performance

Design ideas are based on circumventing the drawbacks of the LLS and subspace algorithms

Recall the drawbacks in LLS TOA-based method:

- The disturbances $\{m_i\}$ are not of same powers but when we use LLS, equal weighting is assumed in each equation

Solution: Proper weighting for each equation, i.e., LLS is modified to linear weighted least squares (LWLS)

At sufficiently small noise condition, we can rewrite the linear equations by including the noise component as:

$$\mathbf{A}\mathbf{x} + \mathbf{p} = \mathbf{b}$$

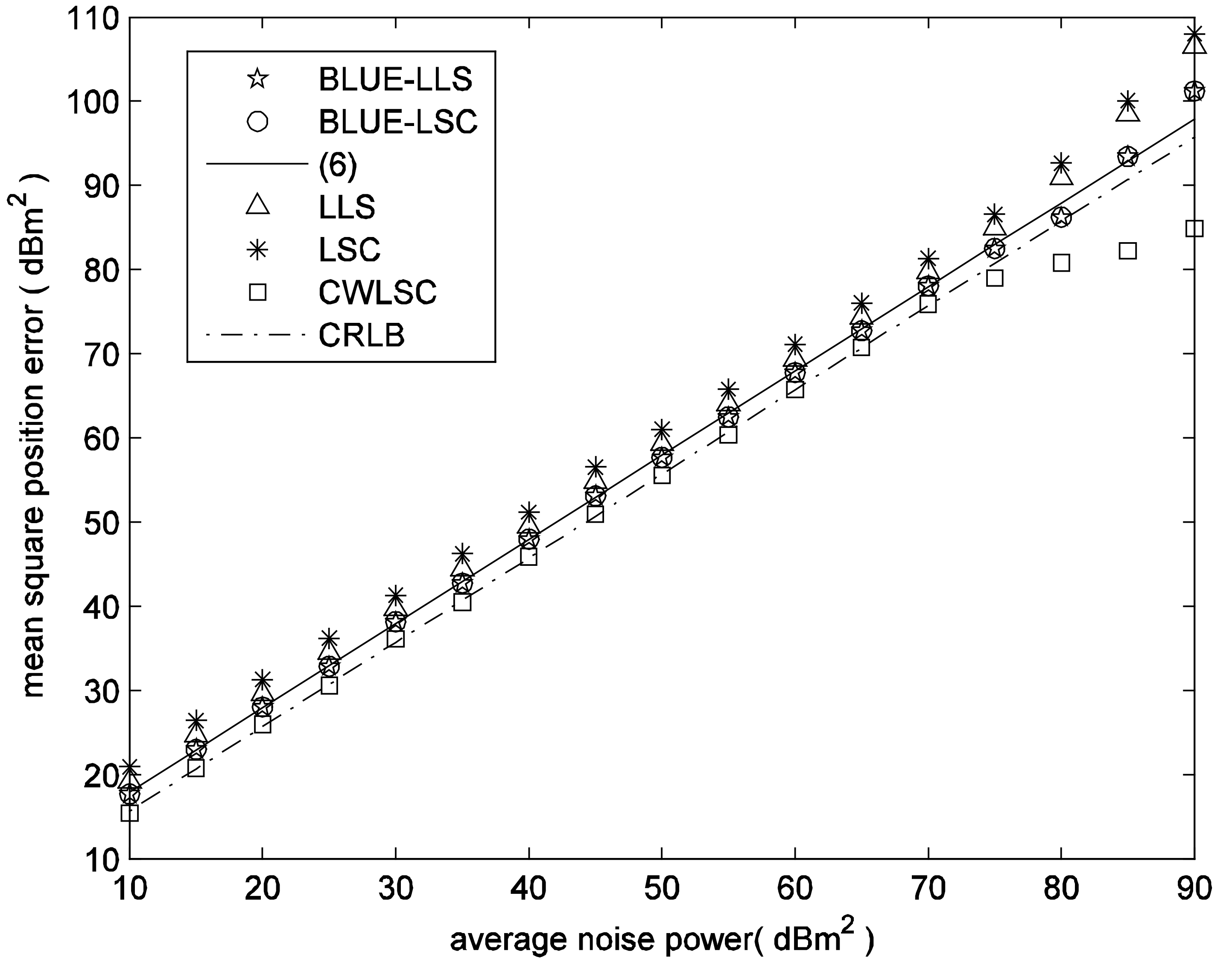
$$\mathbf{A} = \begin{bmatrix} x_1 & y_1 & -0.5 \\ \vdots & \vdots & \vdots \\ x_M & y_M & -0.5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ R \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} -m_1 \\ \vdots \\ -m_M \end{bmatrix} \approx \begin{bmatrix} -d_1 n_1 \\ \vdots \\ -d_M n_M \end{bmatrix}$$

we have $\mathbf{A}\mathbf{x} = E\{\mathbf{b}\}$ which corresponds to linear unbiased model and the best unbiased linear estimator (BLUE) is [5]

$$\hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{C_p}^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{C_p}^{-1} \mathbf{b}$$

where

$$\mathbf{C_p} \approx \begin{bmatrix} d_1^2\sigma^2 & 0 & \cdots & 0 \\ 0 & d_2^2\sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_M^2\sigma^2 \end{bmatrix} \approx \begin{bmatrix} r_1^2\sigma^2 & 0 & \cdots & 0 \\ 0 & r_2^2\sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_M^2\sigma^2 \end{bmatrix}$$




Typical mean square error performance of BLUE

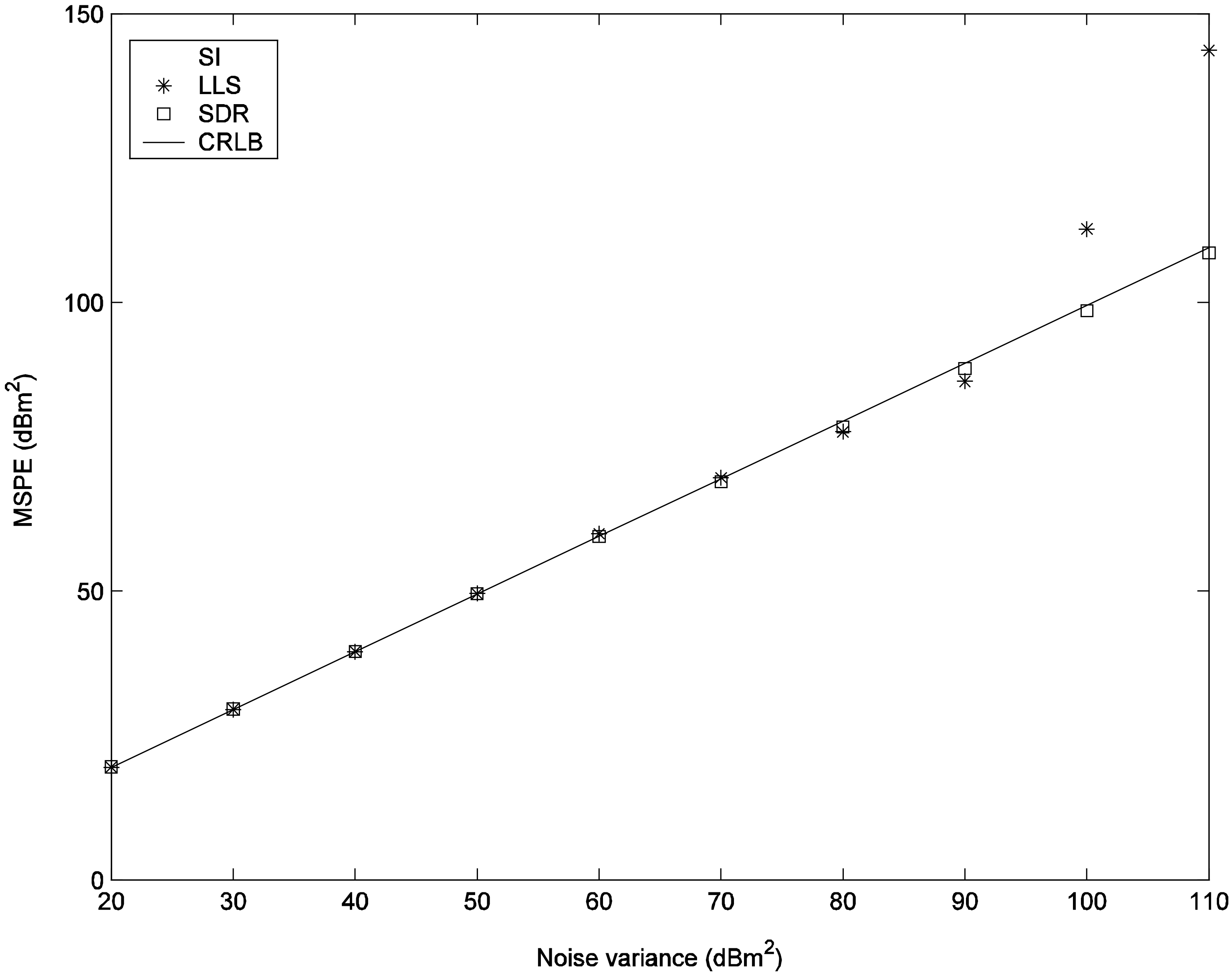
- At sufficiently large noise conditions, the means of $\{m_i\}$ are not close to 0 and this affects LLS accuracy

Solution: Approximate NLS minimization using convex optimization [13]-[16]

Two basic steps are involved:

- Transform the NLS or ML estimator to an equivalent constrained optimization problem
- Relax the constrained optimization problem to a convex optimization problem such that all constraints are convex and linear

The performance of convex optimization approaches ML if the constraints are tight




Typical mean square error performance of semidefinite relaxation

- The introduced variable $R = x^2 + y^2$ is a function of $x$ and $y$ but this information has not been utilized in LLS

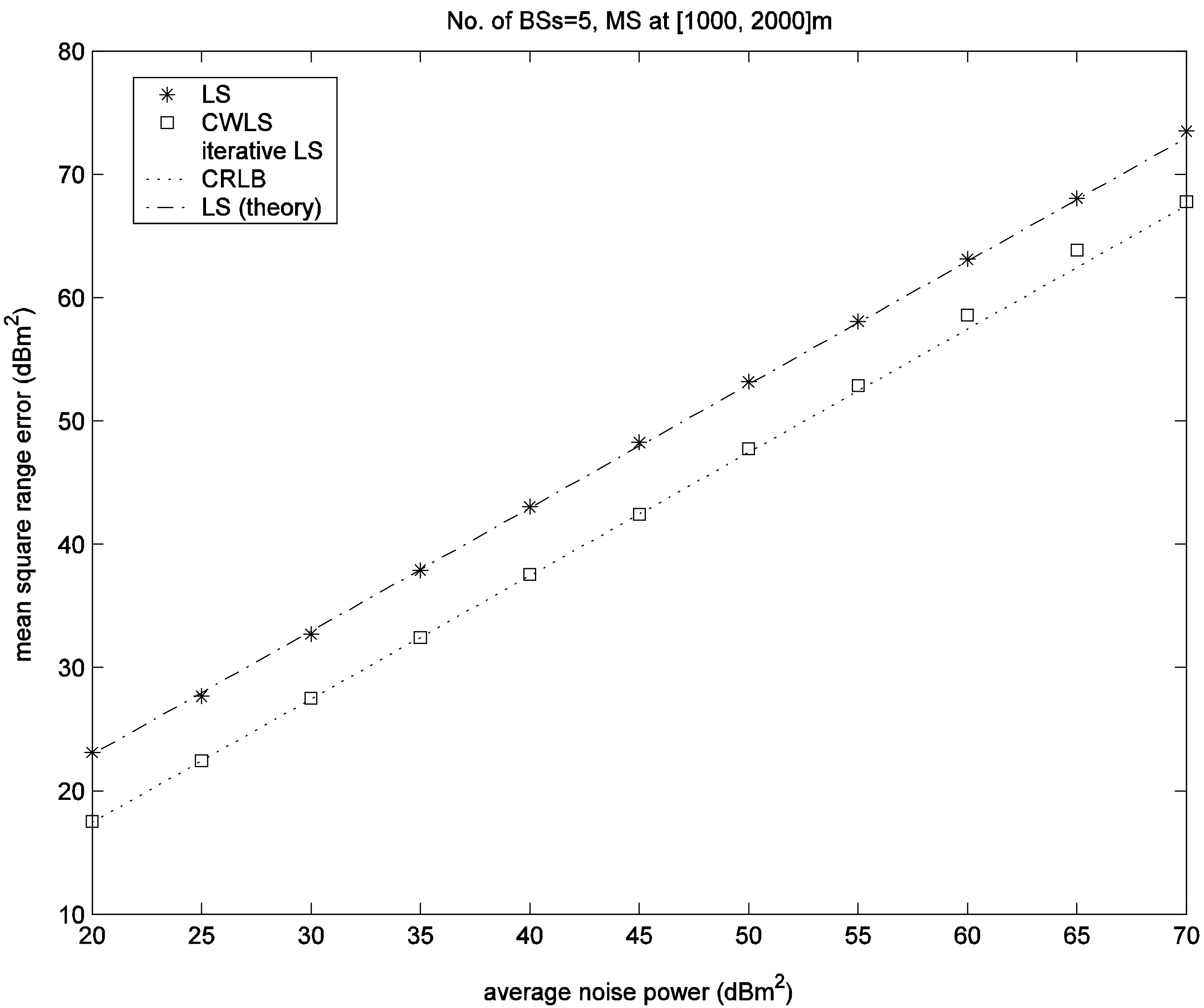
Solution: This information is utilized as a constraint in the minimization, i.e., the position estimate is given by

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \{J(\mathbf{x})\} \quad \text{subject to} \quad R = x^2 + y^2$$

Combining with WLS, the problem

$$\text{Minimize } J(\mathbf{x}) = (\mathbf{A}\mathbf{x} - \mathbf{b})^T \mathbf{C}_{\mathbf{p}}^{-1} (\mathbf{A}\mathbf{x} - \mathbf{b}) \quad \text{subject to} \quad R = x^2 + y^2$$

which is a constrained WLS (CWLS) problem and can be solved by method of Lagrange multipliers [4]




Typical mean square error performance of CWLS method

One drawback in subspace method:

- Noises in $\mathbf{X} \approx \mathbf{U}_s \mathbf{U}_s^T \mathbf{X}$ are not i.i.d. and LS is suboptimal

Solution: Incorporating an optimal weighting matrix in $\mathbf{X} \approx \mathbf{U}_s \mathbf{U}_s^T \mathbf{X}$ [12]

Two basic steps are involved:

- Reorganize $\mathbf{X} \approx \mathbf{U}_s \mathbf{U}_s^T \mathbf{X}$ into the form of $\mathbf{A}\mathbf{x} \approx \mathbf{b}$
- Incorporate the optimal weighting matrix in $\mathbf{A}\mathbf{x} \approx \mathbf{b}$

# Performance Analysis [17]

We start with analyzing the bias and mean square error (MSE) in estimation of a scalar $x$

$$\mathbf{r} = \mathbf{f}(x) + \mathbf{w}$$

Suppose its estimate is obtained by minimizing a differentiable cost function constructed from $\mathbf{r}$, $J(x)$:

$$\hat{x} = \arg\min_{x} J(x)$$

This implies

$$\left.\frac{dJ(x)}{dx}\right|_{x=\hat{x}} = J'(\hat{x}) = 0$$

At small estimation error conditions, $\hat{x}$ is close to $x$. Applying Tayler series expansion yields:

$$J'(\hat{x}) \approx J'(x) + (\hat{x} - x)J''(x)$$

If $J''(x)$ is sufficiently smooth around $x$, then

$$J''(x) \approx E\{J''(x)\}$$

Hence

$$0 = J'(\hat{x}) \approx J'(x) + (\hat{x} - x)E\{J''(x)\} \Rightarrow \hat{x} - x \approx -\frac{J'(x)}{E\{J''(x)\}}$$

$$\Rightarrow \mathrm{bias}(\hat{x}) = E\{\hat{x}\} - x \approx -\frac{E\{J'(x)\}}{E\{J''(x)\}}$$

Similarly,

$$\mathrm{MSE}(\hat{x}) = E\{(\hat{x} - x)^2\} \approx \frac{E\{(J'(x))^2\}}{(E\{J''(x)\})^2}$$

For estimation of a vector $\mathbf{x}$ from minimizing $J(\mathbf{x})$, the formulas are generalized as follows:

$$\mathbf{0} = \frac{\partial J(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \approx \frac{\partial J(\mathbf{x})}{\partial \mathbf{x}} + \frac{\partial^2 J(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}^T} (\hat{\mathbf{x}} - \mathbf{x})$$

$$\Rightarrow -\boldsymbol{\nabla}(J(\mathbf{x})) \approx \mathbf{H}(J(\mathbf{x}))\,(\hat{\mathbf{x}} - \mathbf{x})$$

and

$$\mathbf{H}(J(\mathbf{x})) \approx E\{\mathbf{H}(J(\mathbf{x}))\}$$

where $\boldsymbol{\nabla}(J(\mathbf{x}))$ is the gradient vector and $\mathbf{H}(J(\mathbf{x}))$ is the Hessian matrix:

$$\boldsymbol{\nabla}(J(\mathbf{x})) = \left[ \frac{\partial J(\mathbf{x})}{\partial x_1} \quad \frac{\partial J(\mathbf{x})}{\partial x_2} \quad \cdots \quad \frac{\partial J(\mathbf{x})}{\partial x_L} \right]^T$$

$$
\mathbf{H}(J(\mathbf{x})) = \begin{bmatrix} \dfrac{\partial^2 J(\mathbf{x})}{\partial^2 x_1} & \dfrac{\partial^2 J(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 J(\mathbf{x})}{\partial x_1 \partial x_L} \\ \\ \dfrac{\partial^2 J(\mathbf{x})}{\partial x_2 \partial x_1} & \dfrac{\partial^2 J(\mathbf{x})}{\partial^2 x_2} & \cdots & \dfrac{\partial^2 J(\mathbf{x})}{\partial x_2 \partial x_L} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 J(\mathbf{x})}{\partial x_L \partial x_1} & \dfrac{\partial^2 J(\mathbf{x})}{\partial x_L \partial x_2} & \cdots & \dfrac{\partial^2 J(\mathbf{x})}{\partial^2 x_L} \end{bmatrix}
$$

As a result,

$$
\text{bias}(\hat{\mathbf{x}}) = E\{\hat{\mathbf{x}}\} - \mathbf{x} \approx -[E\{\mathbf{H}(J(\mathbf{x}))\}]^{-1} E\{\boldsymbol{\nabla}(J(\mathbf{x}))\}
$$

Similarly, the covariance matrix is:

$$
\mathbf{C}_{\hat{\mathbf{x}}} \approx [E\{\mathbf{H}(J(\mathbf{x}))\}]^{-1} E\{\boldsymbol{\nabla}(J(\mathbf{x}))\boldsymbol{\nabla}^T(J(\mathbf{x}))\}[E\{\mathbf{H}(J(\mathbf{x}))\}]^{-1}
$$

The MSE of $\hat{x}_l$ is given by $(l,l)$ entry of $\mathbf{C}_{\hat{\mathbf{x}}}$

Consider positioning of a source at $\mathbf{x} = [x \ y]^T$ by $N \geq 3$ sensors at known coordinates $\mathbf{x}_n = [x_n \ y_n]^T$, $n = 1, 2, \ldots, N$

Take TOA model as an illustration:

$$r_n = d_n + w_n, \quad n = 1, 2, \ldots, N$$

where $d_n = \sqrt{(x - x_n)^2 + (y - y_n)^2}$ and $w_n \sim \mathcal{N}(0, \sigma_w^2)$ is white

Taking ML or NLS approach as an illustration, the cost function is

$$J(\mathbf{x}) = \sum_{n=1}^{N} \left( r_n - \sqrt{(x - x_n)^2 + (y - y_n)^2} \right)^2$$

To determine the bias and MSE, the steps include:

$$\nabla(J(\mathbf{x})) = -\frac{2}{\sigma_w^2}\begin{bmatrix} \sum_{n=1}^{N} \frac{(r_n - d_n)(x - x_n)}{d_n} \\ \sum_{n=1}^{N} \frac{(r_n - d_n)(y - y_n)}{d_n} \end{bmatrix} \Rightarrow E\{\nabla(J(\mathbf{x}))\} = \mathbf{0}$$

because $E\{r_n\} = d_n$

Similarly,

$$E\{\mathbf{H}(J(\mathbf{x}))\} = \frac{2}{\sigma_w^2}\begin{bmatrix} \sum_{n=1}^{N} \frac{(x - x_n)^2}{d_n^2} & \sum_{n=1}^{N} \frac{(x - x_n)(y - y_n)}{d_n^2} \\ \sum_{n=1}^{N} \frac{(x - x_n)(y - y_n)}{d_n^2} & \sum_{n=1}^{N} \frac{(y - y_n)^2}{d_n^2} \end{bmatrix}$$

As a result,

$$\text{bias}(\hat{\mathbf{x}}) \approx -[E\{\mathbf{H}(J(\mathbf{x}))\}]^{-1} E\{\boldsymbol{\nabla}(J(\mathbf{x}))\} = \mathbf{0}$$

With tedious calculation, we have

$$\mathbf{C}_{\hat{\mathbf{x}}} \approx \sigma_w^2 \begin{bmatrix} \displaystyle\sum_{n=1}^{N} \frac{(x-x_n)^2}{d_n^2} & \displaystyle\sum_{n=1}^{N} \frac{(x-x_n)(y-y_n)}{d_n^2} \\ \displaystyle\sum_{n=1}^{N} \frac{(x-x_n)(y-y_n)}{d_n^2} & \displaystyle\sum_{n=1}^{N} \frac{(y-y_n)^2}{d_n^2} \end{bmatrix}^{-1}$$

which is the inverse of the Fisher information matrix

That is, the estimator is optimum

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