Tracking Time-Vertex Propagation using Dynamic Graph Wavelets

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Graph Signal Processing: Background

Given a weighted undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, W)$, a graph signal is defined as the mapping

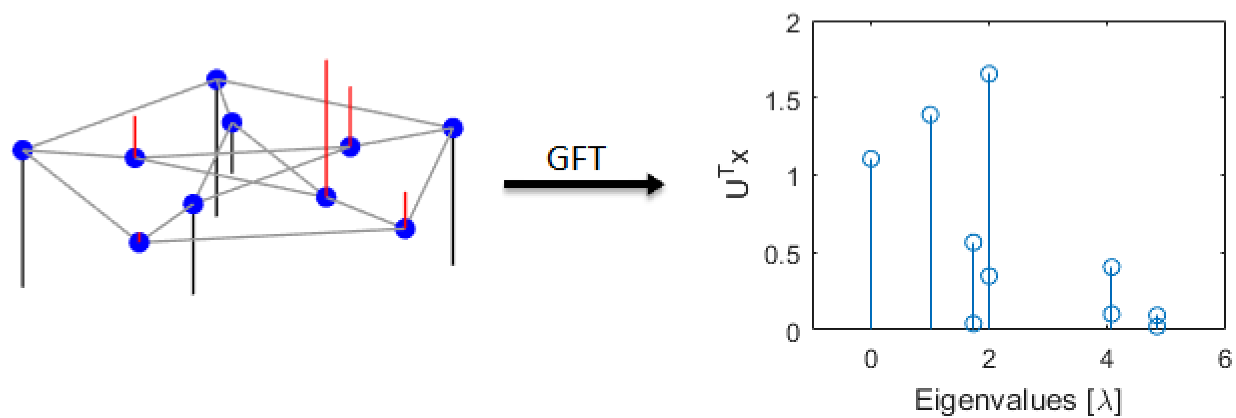
$$x: \mathcal{V} \to \mathbb{R}^{|\mathcal{V}|}$$

The eigendecomposition of the Laplacian of the graph gives us the Fourier modes of the graph

$$L = U \Lambda U^* = \sum_{\ell}^{N} \lambda_\ell u_\ell u_\ell^*$$

GFT: $\tilde{x} = U^* x$

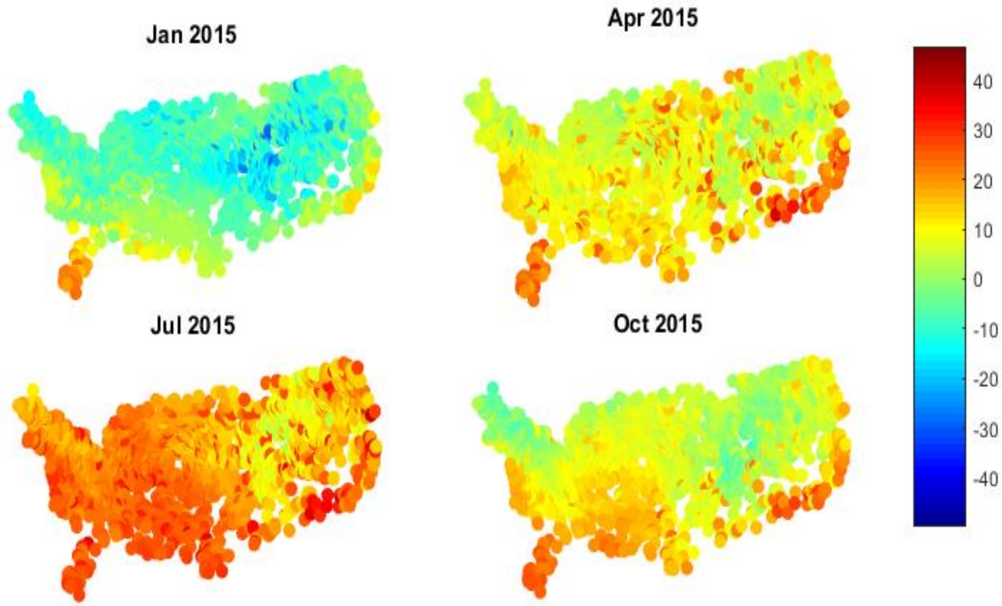
Inverse GFT: $x = U \tilde{x}$



What about time-varying signal on graph?

A great variety of data lying on graph are inherently time-varying

- Weather
- Electrophysiological signals
- Point clouds or video
- Transportation data
- Many others!



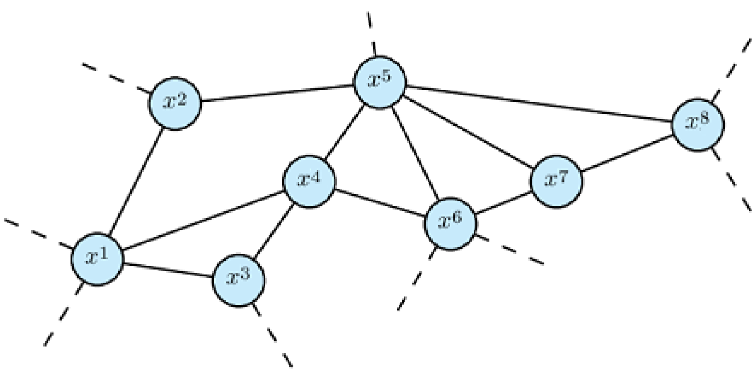
Usually we want to analyze complex dynamic processes and extract relevant information

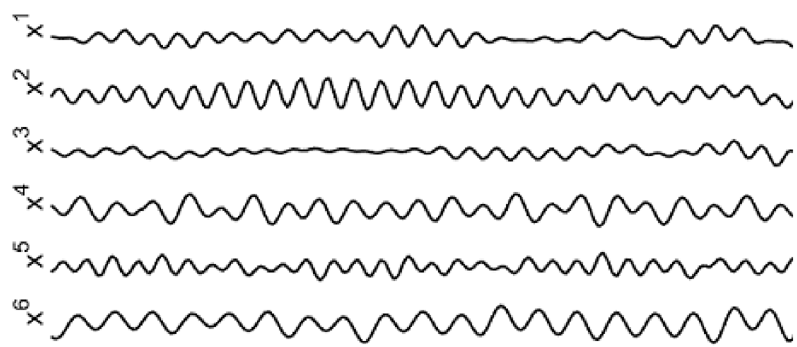
Time-Vertex Signal Processing

Given a weighted undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, W)$ with associated time domain of length T, a time-vertex signal is defined as

$$X \in \mathbb{R}^{|\mathcal{V}| \times T}$$

- Collection of T graph signal $X = [x_1, x_2, \dots, x_T]$
- Collection of N time series $X = [x^1, x^2, \dots, x^N]$



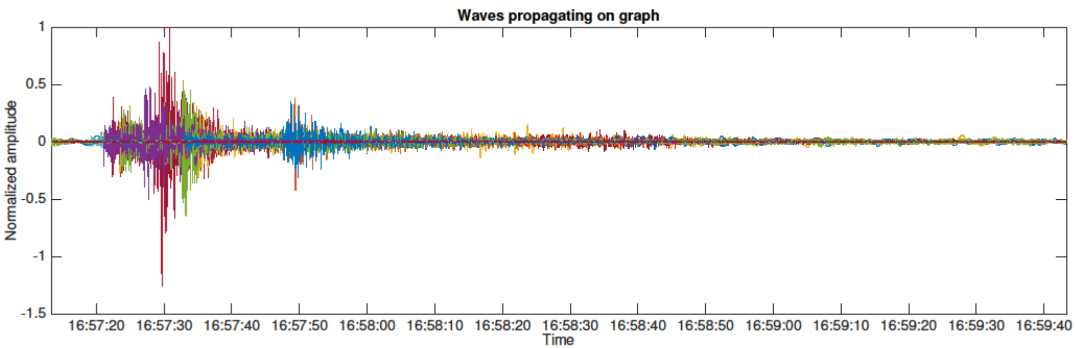


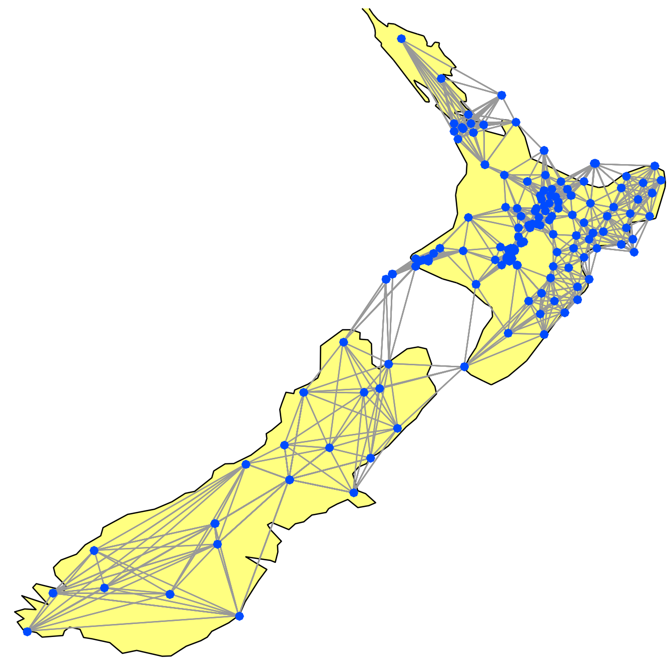
Note: we assume time-invariance of the underlying graph $\mathcal{G}$

Outline

Our work:

1. PDEs as motivation for joint frequency analysis
2. Joint filterbanks and frames based on the wave equation
3. Application to source localization for seismic epicenter estimation





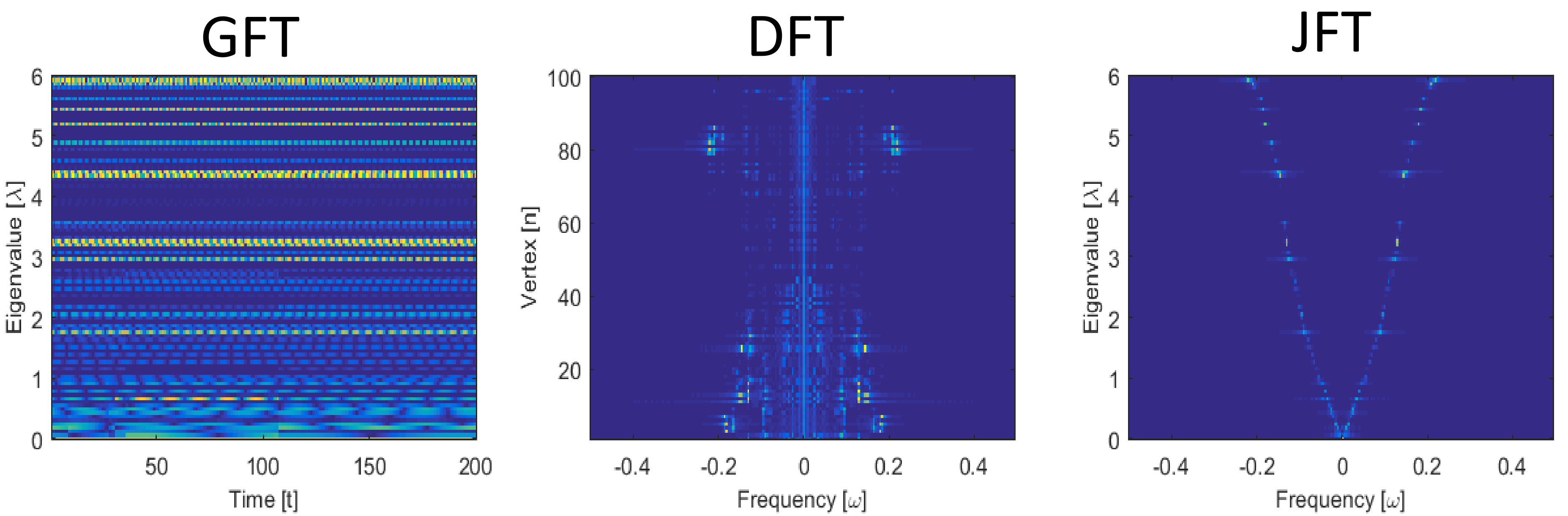
1. PDEs as Motivation to JFT

PDEs as Motivation to JFT

Considering the **Joint Time-Vertex Fourier Transform**

$$\hat{X} = \mathrm{JFT}(X) = U_G^* X \overline{U}_T$$

with U_G^* the Graph Fourier Basis and $\overline{U}_T$ the DFT matrix, we show that JFT domain is able to unveil the structure of the dynamic process evolving on the graph



Wave on Graphs

Let consider the Partial Differential Equation (PDE) associated to the propagation of a wave

$$\frac{\partial^2 x}{\partial t^2} = c^2 \nabla^2 x$$

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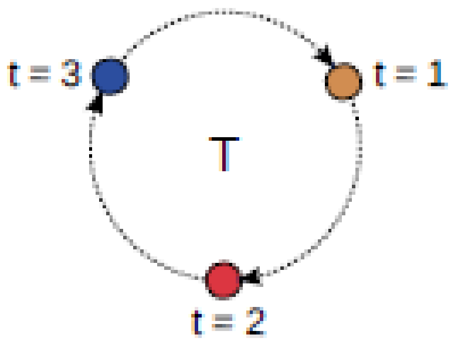
Discretizing both domains

- **Time** $\frac{\partial^2 x}{\partial t^2} = -L_T$



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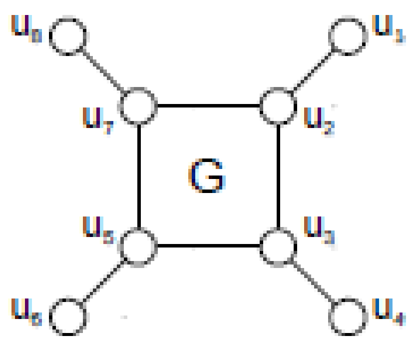
- **Space** $\nabla^2 = -L_G$



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- Space $\nabla^2 = -L_G$

We obtain the **discrete wave equation on the graph** $\mathcal{G}$

$$X L_T = \alpha^2 L_G X$$

where α^2 is related to the speed of propagation of the wave and to the ratio between the sampling frequency in time and space

Wave on Graphs: Solution

Assuming that the solution at time t can be written as a **propagator operator** $K_\alpha(L_G, t)$ applied to the initial condition x_0

$$x_t = K_\alpha(L_G, t) x_0$$

$$X = [K_\alpha(L_G, t) x_0]_{t=1}^{T} = K_\alpha\{x_0\}$$

Substituting and applying the GFT

$$\widetilde{K}_\alpha\{\tilde{x}_0\} L_T = \alpha^2 \Lambda_G \, \widetilde{K}_\alpha\{\tilde{x}_0\}$$

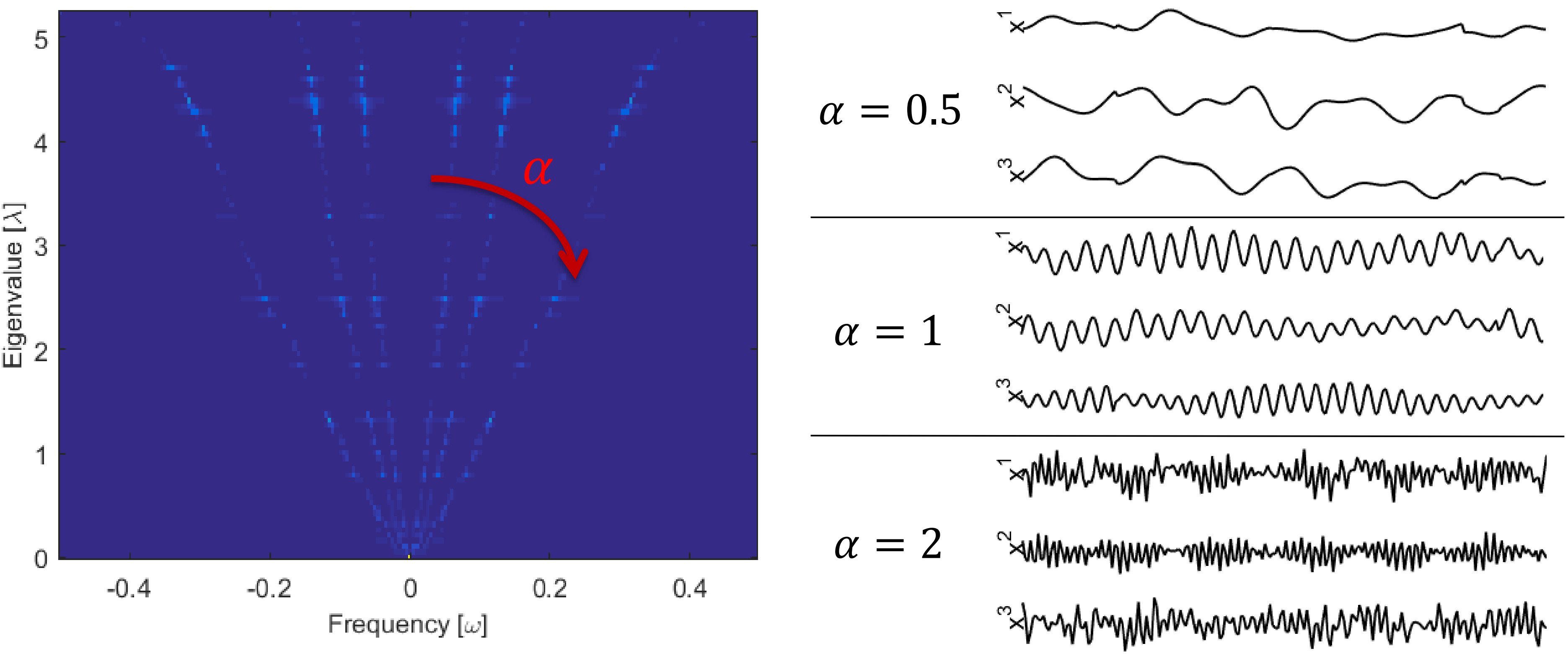
That is an eigenvalue equation with solution

$$\widetilde{K}_\alpha(\lambda_\ell, t) = \cos\left(t \arccos\left(1 - \frac{\alpha^2 \lambda_\ell}{2}\right)\right) = \cos(t\theta_\ell)$$

denoted **Wave Filter**

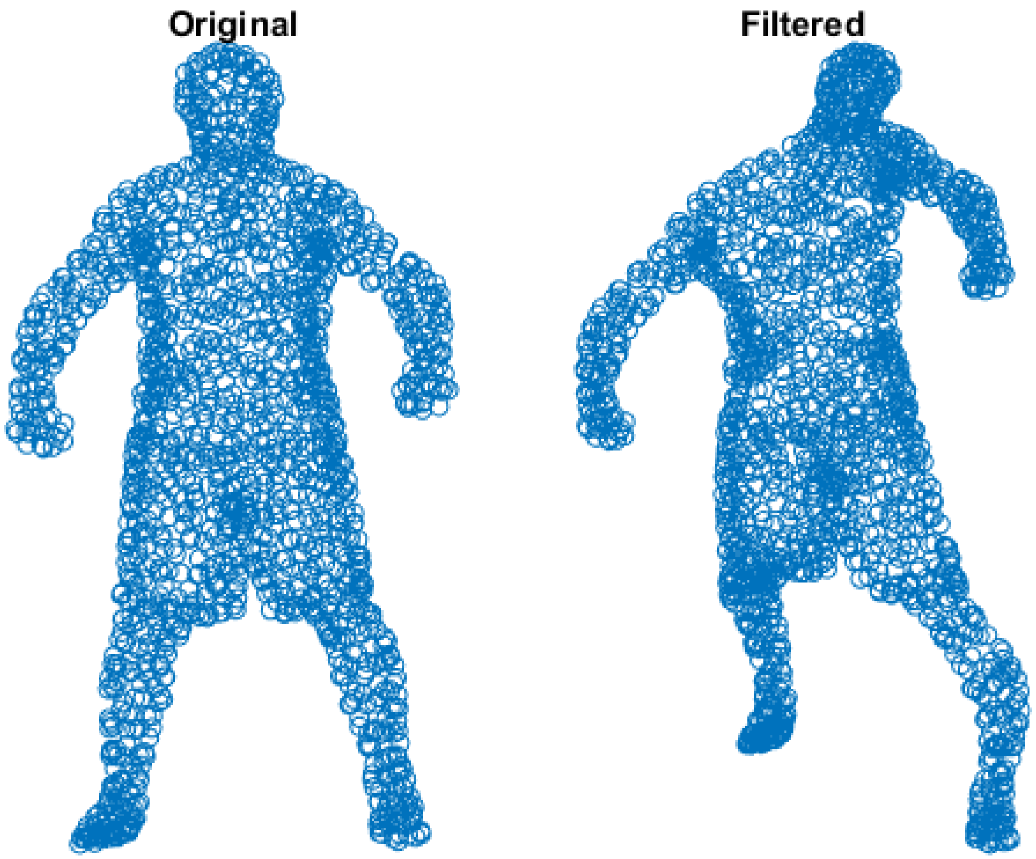
Wave Filter

$$\widetilde{K}_\alpha(\lambda_\ell, t) = \cos(t\theta_\ell) \xrightarrow{DFT} \widehat{K}_\alpha(\lambda_\ell, \omega_k) = \frac{1}{2}\left(\delta(\omega_k + \theta_\ell) + \delta(\omega_k - \theta_\ell)\right)$$



Wave Filter: Application

$$\widetilde{K}_\alpha(\lambda_\ell, t) = \cos\left(t \arccos\left(1 - \frac{\alpha^2 \lambda_\ell}{2}\right)\right)$$



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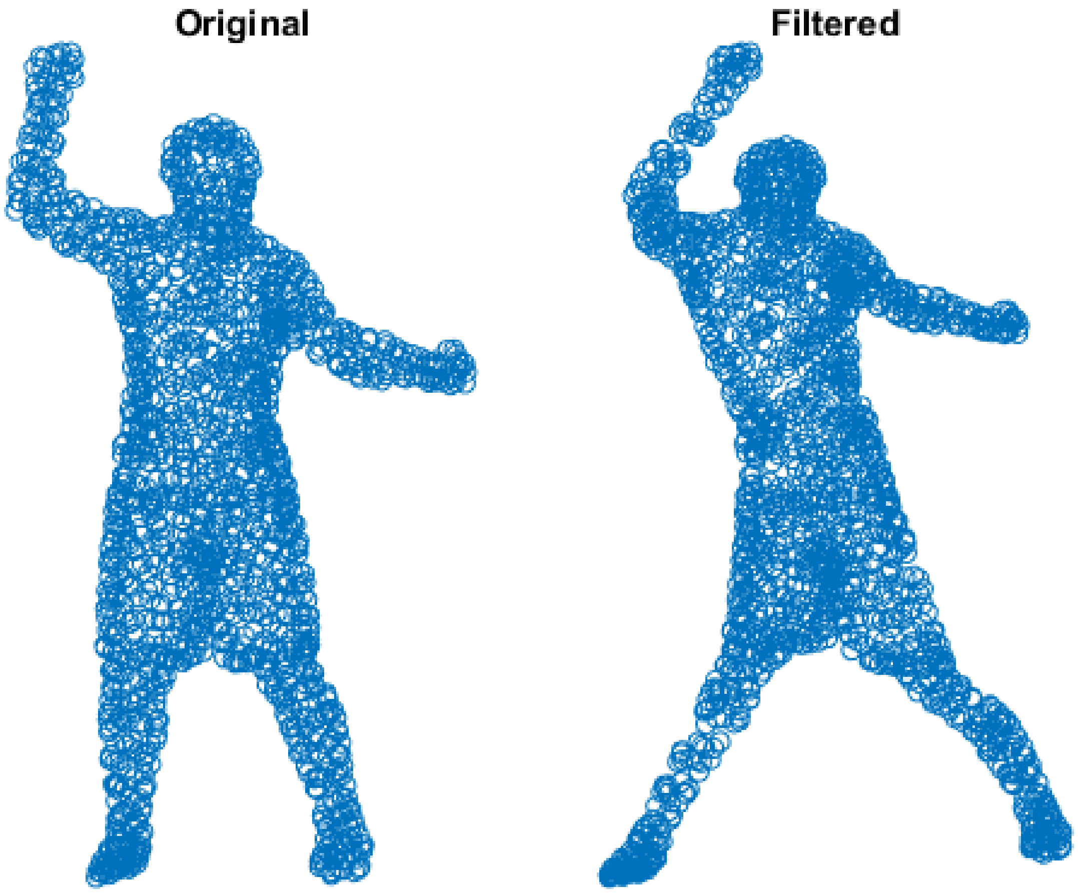

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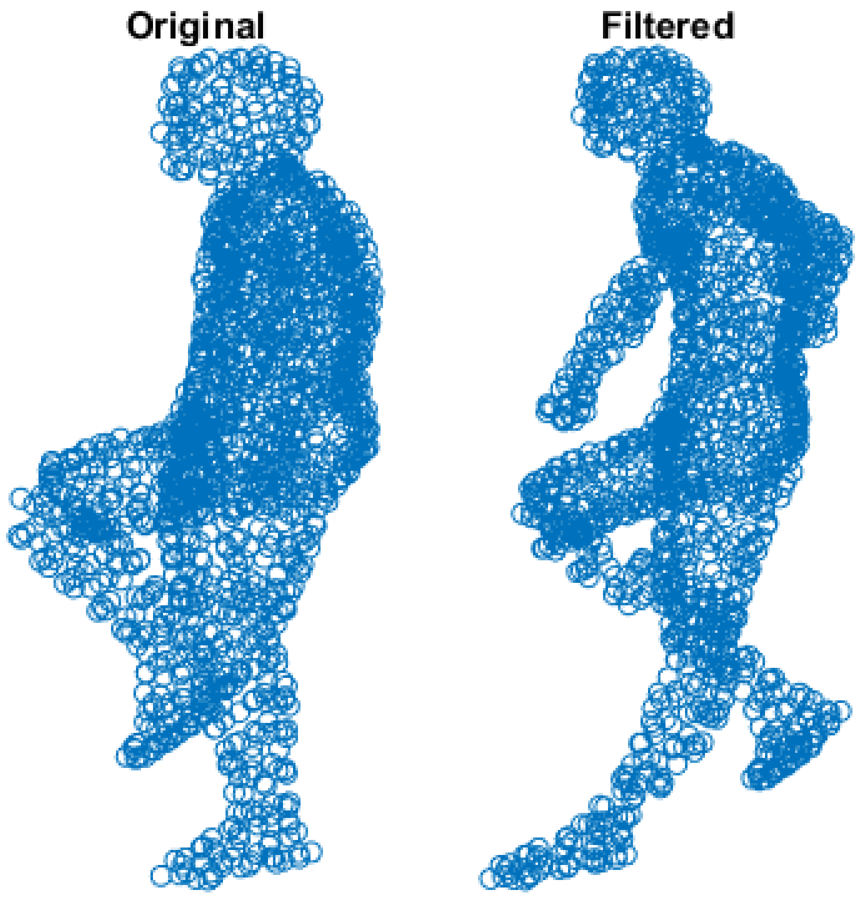
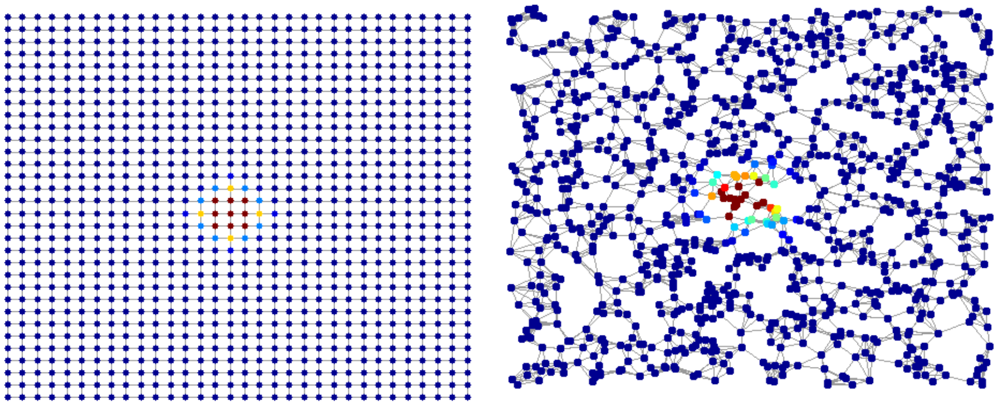


2. Dynamic Graph Wavelets

Dynamic Graph Wavelets

Definition: Dynamic Graph Wavelets (DGW) allow to represent a time-vertex process as combination of propagating signals on the graph $\mathcal{G}$ that evolve in time according to a linear PDE, with parameter α



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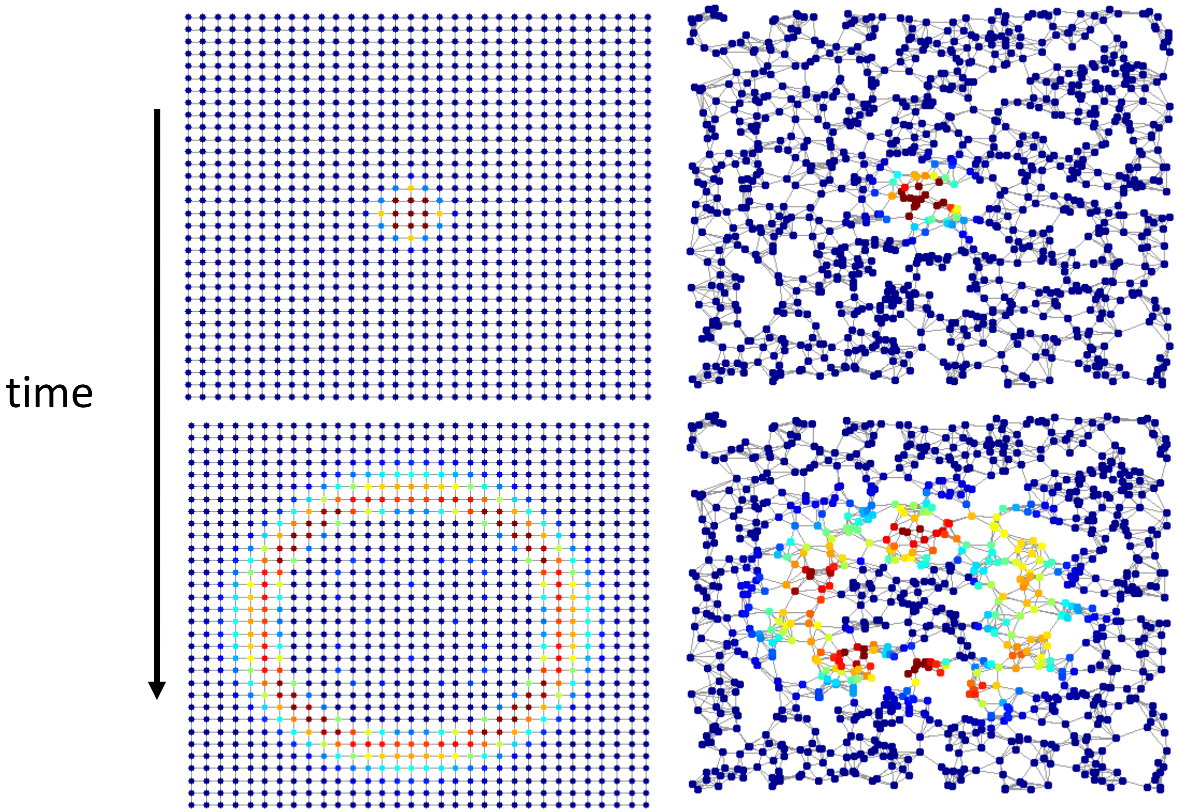


Dynamic Graph Frames

Is possible to find the condition under which the joint time-vertex filterbank forms a **frame** S_h, i.e. the **frame bound** holds

$$A\|X\|_F^2 \leq \|S_h\{X\}\|_F^2 \leq B\|X\|_F^2$$

for any $\|X\|_F^2 > 0$

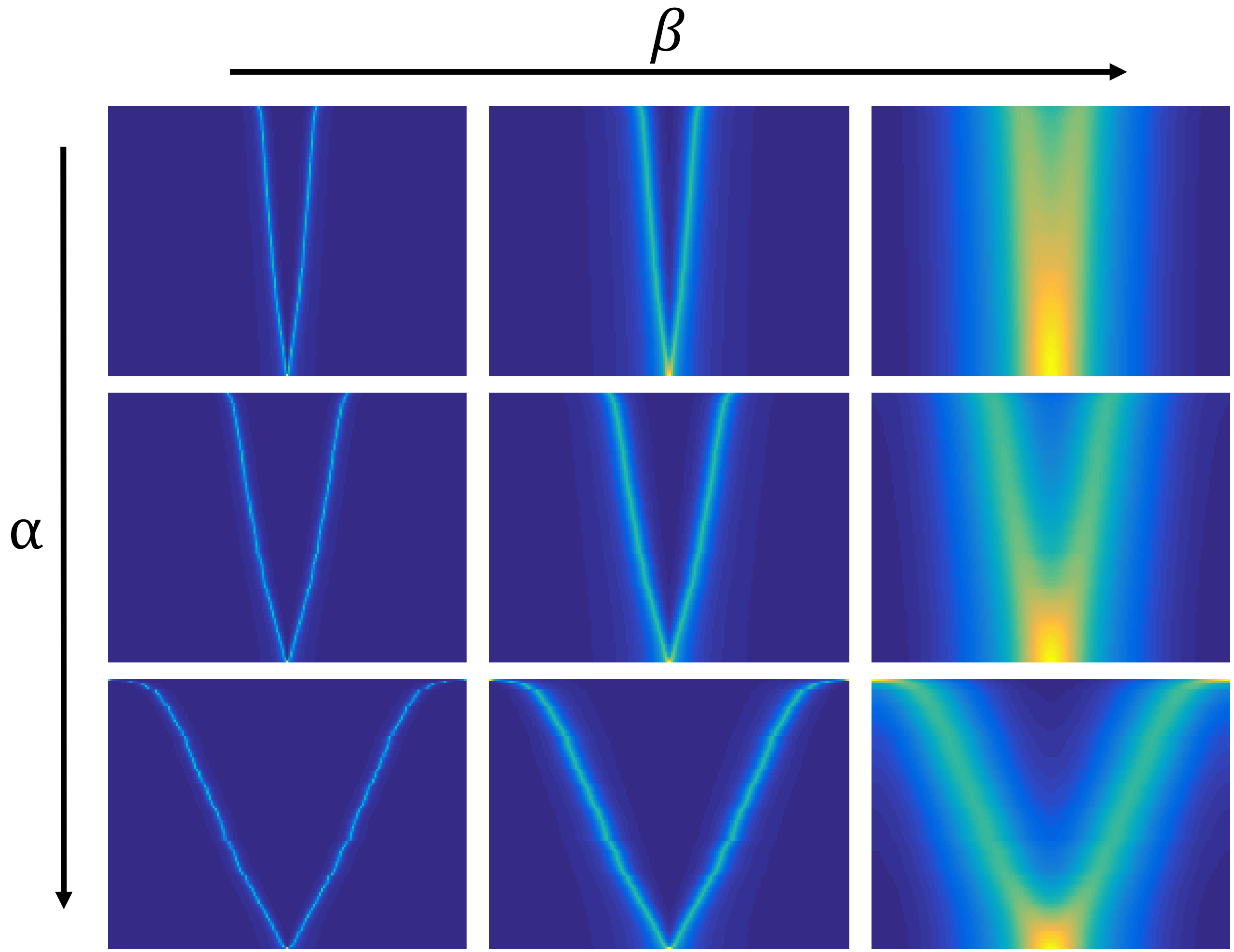
The **Causal Damped Wave** Dynamic Graph Wavelet

$$W_{\alpha,\beta}(\lambda_\ell, t) = \mathrm{H(t)e}^{-\beta t} \cos\left(t \arccos\left(1 - \frac{\alpha^2 \lambda_\ell}{2}\right)\right)$$

forms a **frame** if $\beta > 0$

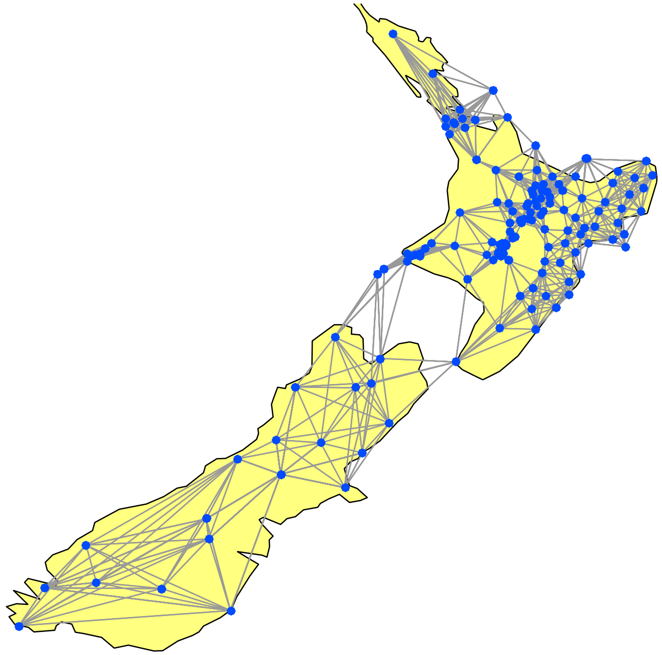
Dynamic Graph Frame: Example

$$W_{\alpha,\beta}(\lambda_\ell, t) = \mathrm{H(t)e}^{-\beta t} \cos\left(t \arccos\left(1 - \frac{\alpha^2 \lambda_\ell}{2}\right)\right)$$



3. Source Localization

Seismic Epicenter Estimation



- We analyze waveforms recorded by seismic stations geographically distributed in New Zealand connected to the GeoNet Network
- We want to estimate the epicenter of the seismic event using the Damped Wave DGW
- We solve the following optimization problem

$$\underset{c}{\operatorname{argmin}} \left\| S_W^T c - Y \right\|_F^2 + \gamma \|c\|_1$$

S_g is the frame operator of a damped wave filterbank $\{W_{\alpha,\beta}\}$ with different speeds α and fixed damping factor β

Seismic Epicenter Estimation

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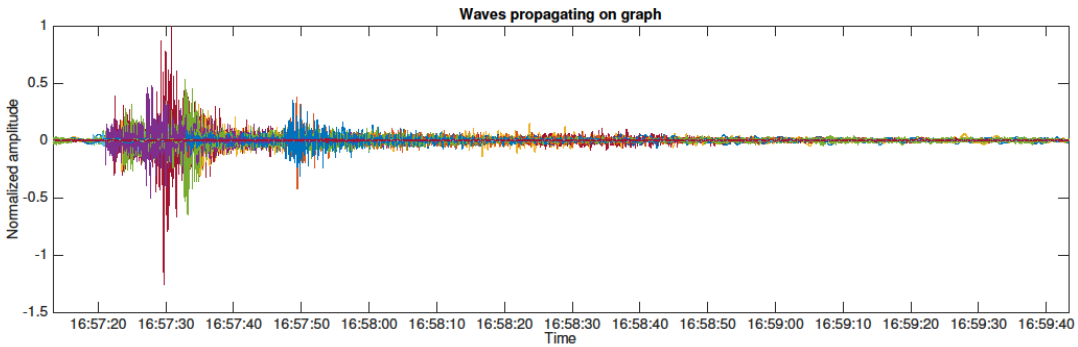
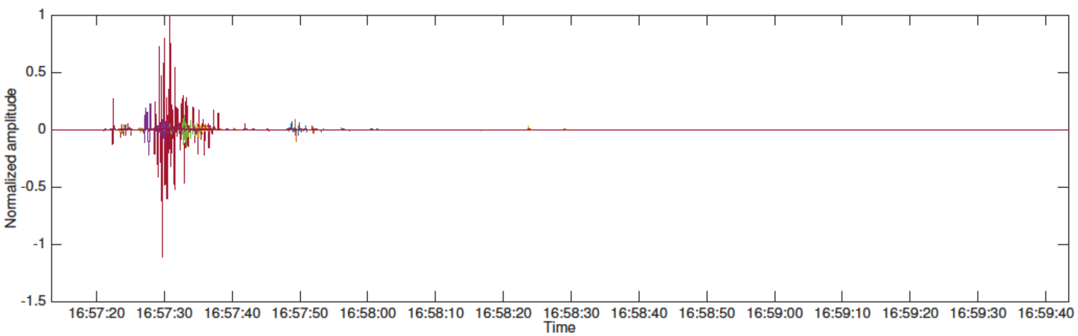
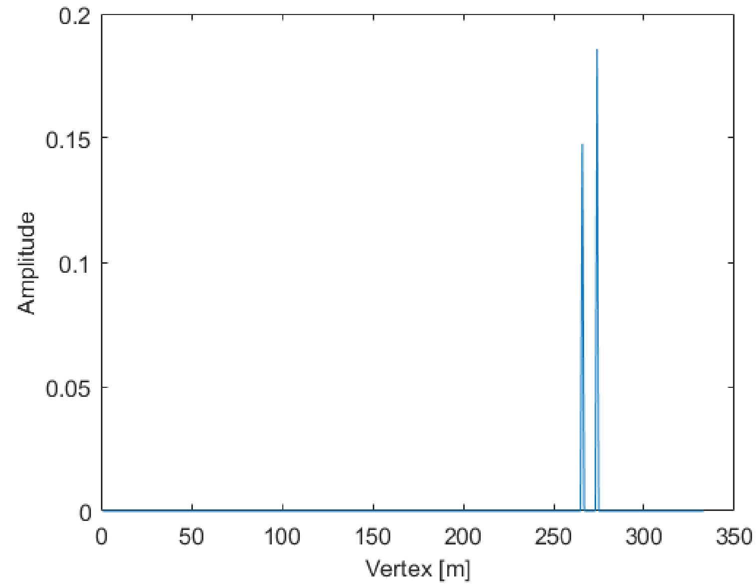
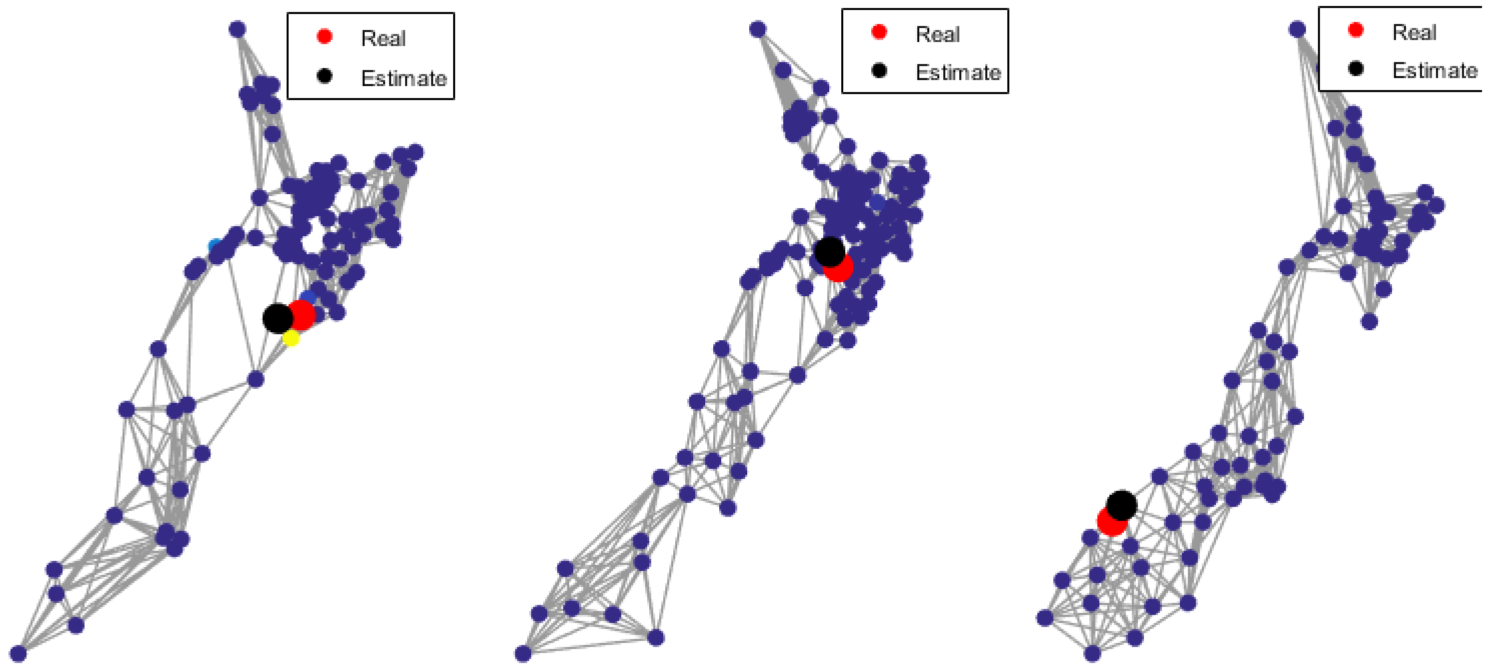






Seismic Epicenter Estimation: Results



- We estimate the epicenter for a set of 40 seismic events
- Comparison with estimate based on sole signal amplitude shows a threefold improvement in average
- Our method is also robust to noise

Conclusion

- PDEs provide insight on Joint Time-Vertex Fourier Analysis and motivate this framework
- Joint time-vertex filters based on PDEs, in particular the wave equation, can be used to analyze and process complex signals in the joint domain
- Dynamic Graph Wavelets are able to retrieve or process information on dynamic phenomena taking place over graphs that can be modeled approximately using linear PDEs

Thank you!