TOMOGRAPHIC RECONSTRUCTION OF ATMOSPHERIC DENSITY WITH MUMFORD-SHAH FUNCTIONALS

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Introduction

Background

Space-based remote sensing of solar radiation scattered by hydrogen (H) atoms is often used to infer the density of Earth's uppermost atmosphere based on an assumed functional form for the global density distribution. Here, we present a alternative approach to conventional parametric estimation that is based instead on tomographic inversion of satellite observations of scattered ultraviolet emission at 121.6 nm.

Forward Model

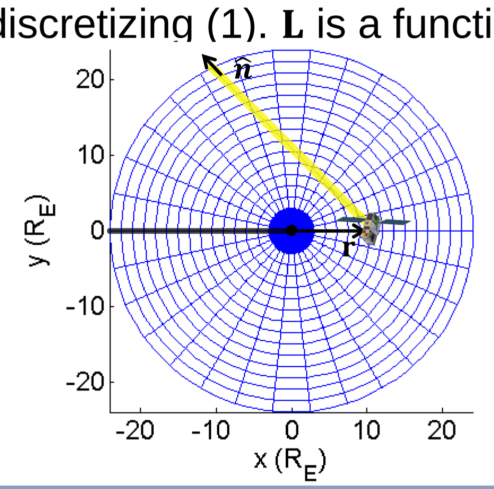
Atmostpheric H emission intensity y, as measured from planetocentric position $\boldsymbol{r}$ along a line-of-sight (LOS) in the direction $\hat{\boldsymbol{n}}$, is related to the unknown H density ρ in terms of distance $l \equiv |\boldsymbol{r}' - \boldsymbol{r}|$ (measured in R_E, radius of earth) along LOS as follows:

$$y(\mathbf{r}, \hat{\boldsymbol{n}}) \propto \int_0^{l_\infty(\hat{\boldsymbol{n}})} \rho(\boldsymbol{r}')dl \quad (1)$$

Utilizing standard basis functions in polar coordinates, the 2-D plane is discretized into **polar rectangles**, and assuming the density distribution is **constant** within each polar rectangle, the following can be derived:

$$\mathbf{y} = \mathbf{Lx} + \mathbf{w} \qquad (2)$$

- $\mathbf{y}, \boldsymbol{w} \in \mathbb{R}^I$, I observed emission intensity and Poisson distributed additive noise
- $\mathbf{x} \in \mathbb{R}^J$, H density in J polar rectangle
- $\mathbf{L} \in \mathbb{R}^{I \times J}$, observation matrix constructed from discretizing (1). $\mathbf{L}$ is a function of $\hat{\boldsymbol{n}}, \mathbf{r}$



2D Demonstration of variables defined above in a discretized planetecentric coordinate. Satellite has position vector $\boldsymbol{r}$ with look angle $\hat{\boldsymbol{n}}$ and LOS (yellow)

Aim

Due to the nature of limited data and errors in measurement, the inverse problem is ill-conditioned. Previous techniques include:

- **Tikhonov**: $J(\mathbf{x}, \mathbf{s}) = \underbrace{||\boldsymbol{y} - \boldsymbol{Lx}||^2}_{Data\ Fidelity} + \underbrace{\alpha_1^2||\boldsymbol{Dx}||^2}_{Regularizer}$
  - Linear optimization problem
  - Over smooth solution, edges not preserved
- **Total Variation**: $J(\mathbf{x}, \mathbf{s}) = \underbrace{||\boldsymbol{y} - \boldsymbol{Lx}||^2}_{Data\ Fidelity} + \underbrace{\alpha_1^2|\boldsymbol{Dx}|^2}_{Regularizer}$
  - Well known for edge preservation
  - Edges are not limited, can appear randomly
  - Cost function is non-differentiable

Method

Mumford-Shah Functionals

$$J(\boldsymbol{u}) = \int_\Omega (\boldsymbol{u} - \boldsymbol{f})^2 dA + \alpha_1^2 \int_{\Omega \backslash \Gamma} |\nabla \boldsymbol{u}|^2 dA + \alpha_2^2 |\Gamma|^2$$

- $\Omega, \Gamma, \boldsymbol{u}, \boldsymbol{f}$: Image domain, boundary, and fields
- Originally used for Image Segmentation
- Produces **piece-wise smooth** images
- The functional is non-differentiable

Ambrosio–Tortorelli Approximation

$$J(\mathbf{x}, \mathbf{s}) = \underbrace{||\boldsymbol{y} - \boldsymbol{Lx}||^2}_{Data\ Fidelity} + \underbrace{\alpha_1^2||\boldsymbol{Dx}||_{\boldsymbol{W}_s}^2 + \alpha_2^2||\boldsymbol{Ds}||_2^2 + \alpha_3^2||\boldsymbol{s}||_2^2}_{Regularizer}$$

- $\boldsymbol{D}$: Discrete differencing matrix
- $\boldsymbol{s}$: Edge field corresponding to each "pixel"
- $\boldsymbol{W}_s$: Weighting matrix according to edge field $\boldsymbol{s}$
- Allowed for Maximum-*a-posteriori* estimation interpretation
- Non-linear optimization problem due to $\boldsymbol{W}_s$

Optimization Steps

Step 1: Initialization
- Initial guess $\boldsymbol{x_0}$, $\boldsymbol{s_0}$
- Compute $\boldsymbol{W}_s = diag(1 - \boldsymbol{s}^2)$

Step 2.1, Fix $\boldsymbol{s}$, find minimizer $\hat{\boldsymbol{x}}$ that satisfies:

$$(\boldsymbol{L}^T\boldsymbol{L} + \alpha_1^2 \boldsymbol{D}^T\boldsymbol{W}_s\boldsymbol{D})\hat{\boldsymbol{x}} = \boldsymbol{L}^T\boldsymbol{y}$$

Compute $\boldsymbol{W}_x = diag(\alpha_1^2(\boldsymbol{Dx})^2 + \alpha_3^2)$

Compute $\boldsymbol{z} = \boldsymbol{W}_x^{-1}(\alpha_1^2(\boldsymbol{Dx})^2)$

Step 2.2, Fix $\boldsymbol{x}$, find minimizer $\hat{\boldsymbol{s}}$ that satisfies:

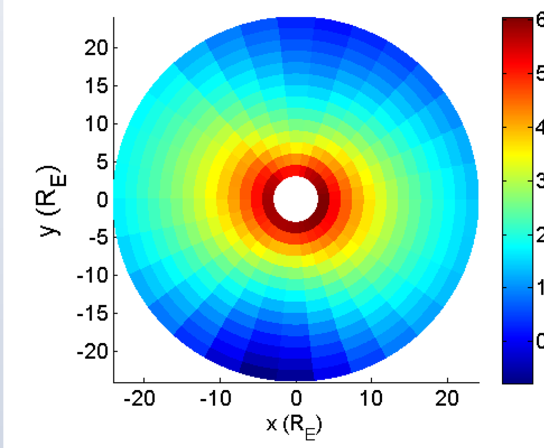
$$(\boldsymbol{W}_x + \alpha_2^2 \boldsymbol{D}^T\boldsymbol{D})\hat{\boldsymbol{s}} = \boldsymbol{W}_x\boldsymbol{z}$$

Compute $\boldsymbol{W}_s$

Step 3: Convergent Solution
- Smooth edge preservation
- Model independent structure of H density

Results

Synthetic Model



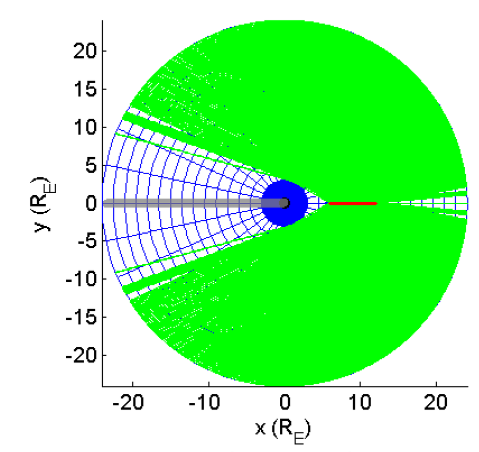
Simulated Hydrogen field with smooth depletion region

- Spherical Harmonics (2D)

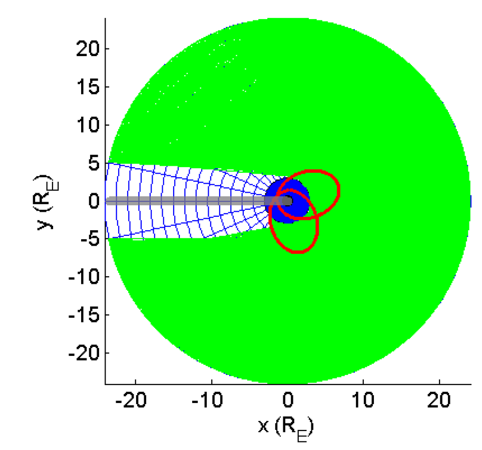
$$\rho\left(r_k, \theta = \frac{\pi}{2}, \phi\right) = \mathcal{N}(r_k)\sqrt{4\pi}\sum_{l=0}^{3}\sum_{m=0}^{l}([A_{lm}(r_k)\cos(m\phi) + B_{lm}(r_k)\sin(m\phi)]Y_{lm}(\theta))$$

- Realistic polar 10% - 20% depletion region
- Angular Res.: $\frac{\pi}{15}$
- Radial Res.: 1.5 R_E

Viewing Geometries



Case A: *Linear Trajectory*

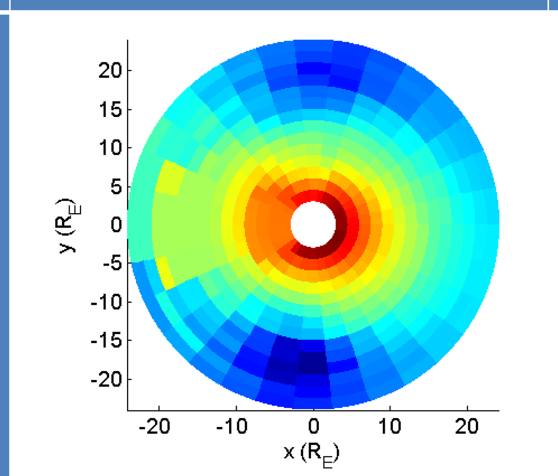
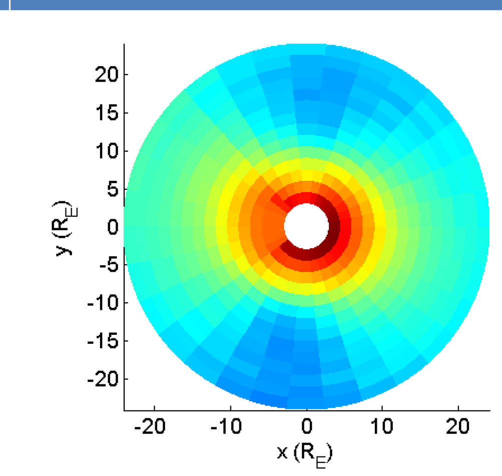
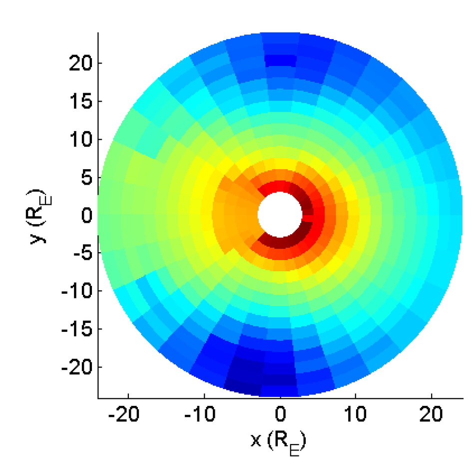
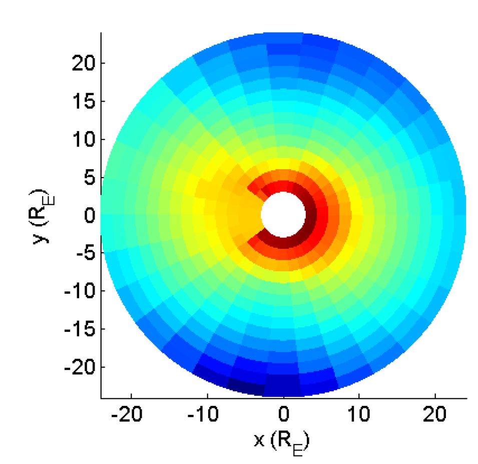


Case B: *Elliptical Trajectory*

Illustration of Line-of-sight coverage of the 2-D field (green) from the measurement positions along the satellite trajectory (red)

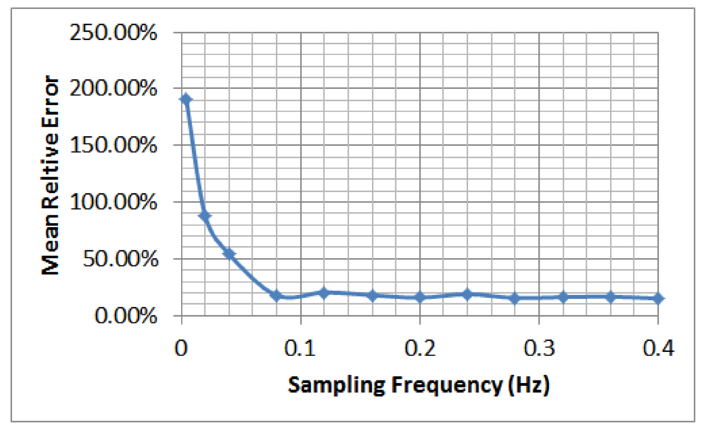
Observations along LOS that transit very near the earth (blue) or through its shadow (grey) are neglected due to complicated scattering physics occurring there that invalidates the forward model

Reconstruction Results

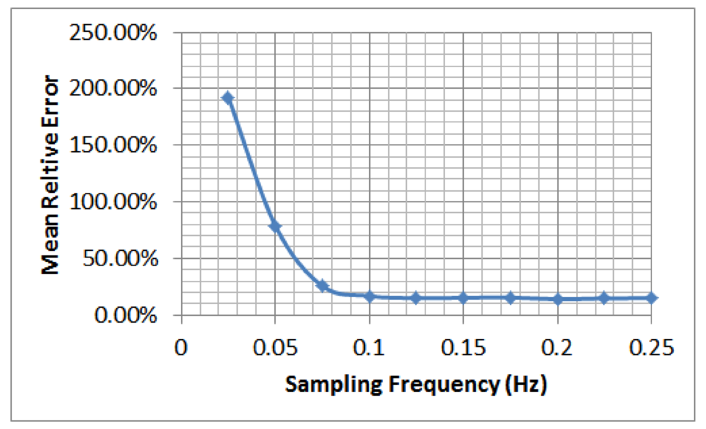
	Case A	Case B
Total Variation	 • **Non-smooth edges**	 • **Large relative error in depletion region**
Mumford-Shah	 • **Relatively accurate prediction result**	 • **Able to preserve the depletion region**

In addition, the reconstruction quality of a single "pixel" depends on:

- Amount of LOS passing through
- Magnitude of atmospheric density



Relative error in the density reconstructions using MS functionals for the linear CASE A viewing geometry as a function of sampling frequency



Relative error in the density reconstructions using MS functionals for the elliptical CASE B viewing geometry as a function of sampling frequency

Summary & Future Work

1. We proposed the technique of **tomographic inversion** to make sense of the H density distribution, and the result is model-independent.
2. We proposed using Mumford-Shah functionals as the regularizer, which allows "smart" **edge preservation** while maintaining **smoothness**. It also produces superior results comparing to Total Variation method.
3. We will aim at developing optimal regularization. **parameter selection** methods and investigating the fundamental limits of reconstruction **resolution**.

Acknowledgments

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Reference

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2. L. Ambrosio and V. M. Tortorelli, "Approximation of functional depending on jumps by elliptic functional via t-convergence," Communications on Pure and Applied Mathematics, vol. 43, no. 8, pp. 999–1036, 1990.