Moment Relaxations of Optimal Power Flow Problems: Beyond the Convex Hull

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Optimal Power Flow (OPF) Problem

- Optimization used to determine system operation
  - Minimize generation cost while satisfying physical laws and engineering constraints
  - Yields generator dispatches, line flows, etc.

- Many related problems:
  - State estimation, unit commitment, transmission switching, contingency analysis, voltage stability margins, etc.

> "Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF."
> R.P. O'Neill, Chief Economic Advisor, US Federal Energy Regulatory Commission, 2013.

Classical OPF Problem

$$\min_{V_d, V_q} \quad \sum_{k \in \mathcal{G}} \left(c_{2k} P_{Gk}^2 + c_{1k} P_{Gk} + c_{0k} \right)$$

Generation Cost

$$\text{subject to} \quad P_{Gk}^{\min} \leq P_{Gk} \leq P_{Gk}^{\max}$$

Engineering Constraints

$$Q_{Gk}^{\min} \leq Q_{Gk} \leq Q_{Gk}^{\max}$$

$$\left(V_k^{\min}\right)^2 \leq V_{dk}^2 + V_{qk}^2 \leq \left(V_k^{\max}\right)^2$$

$$|S_{lm}| \leq S_{lm}^{\max}$$

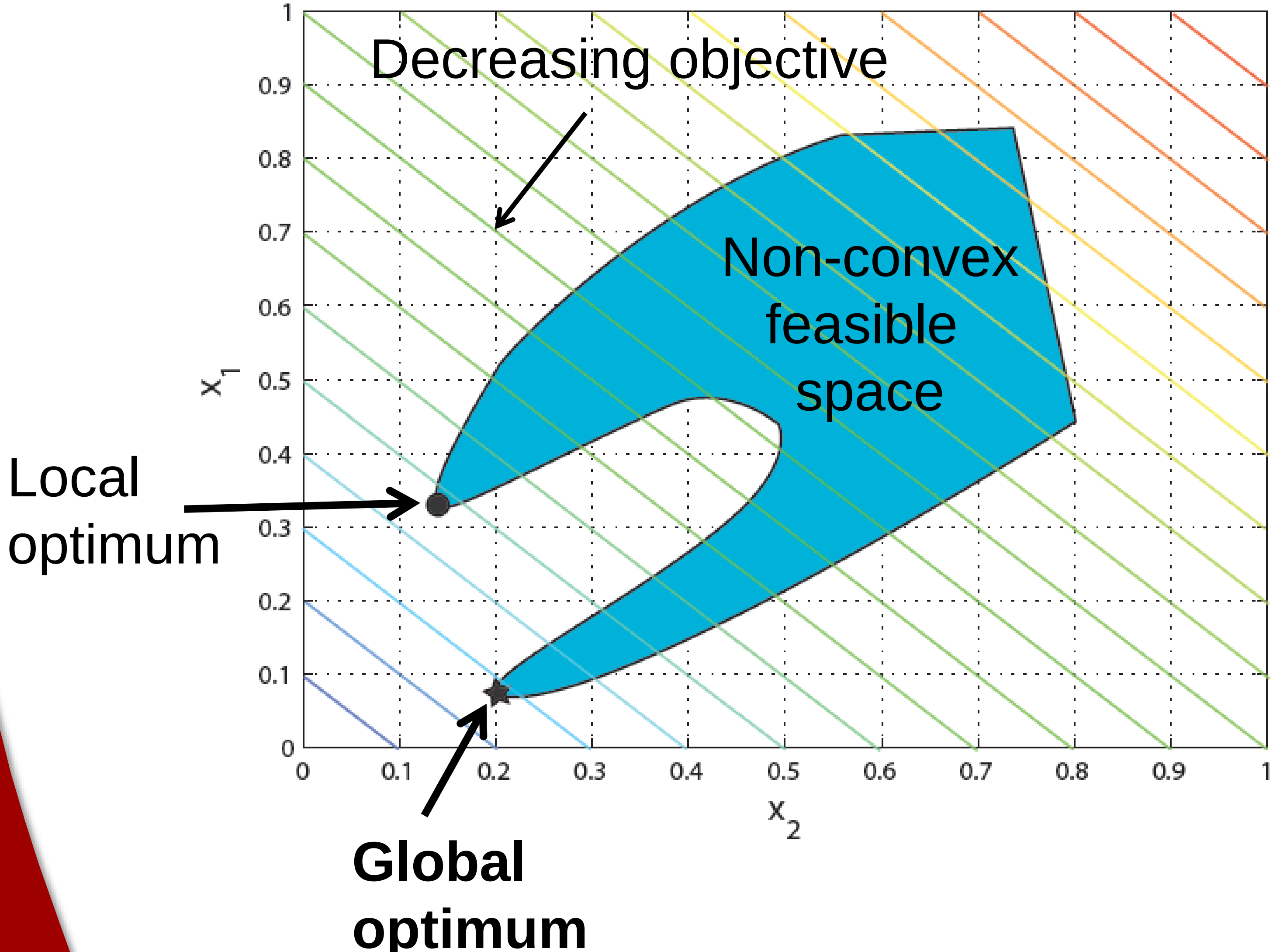
Physical Laws

$$P_{Gk} - P_{Dk} = V_{dk} \sum_{i=1}^{n} (\mathrm{G}_{ik} V_{di} - \mathrm{B}_{ik} V_{qi}) + V_{qk} \sum_{i=1}^{n} (\mathrm{B}_{ik} V_{di} + \mathrm{G}_{ik} V_{qi})$$

$$Q_{Gk} - Q_{Dk} = V_{dk} \sum_{i=1}^{n} (-\mathrm{B}_{ik} V_{di} - \mathrm{G}_{ik} V_{qi}) + V_{qk} \sum_{i=1}^{n} (\mathrm{G}_{ik} V_{di} - \mathrm{B}_{ik} V_{qi})$$

Rectangular voltage coordinates: $V = V_d + jV_q, \; V_d, V_q \in \mathbb{R}^n$

Convex Relaxation



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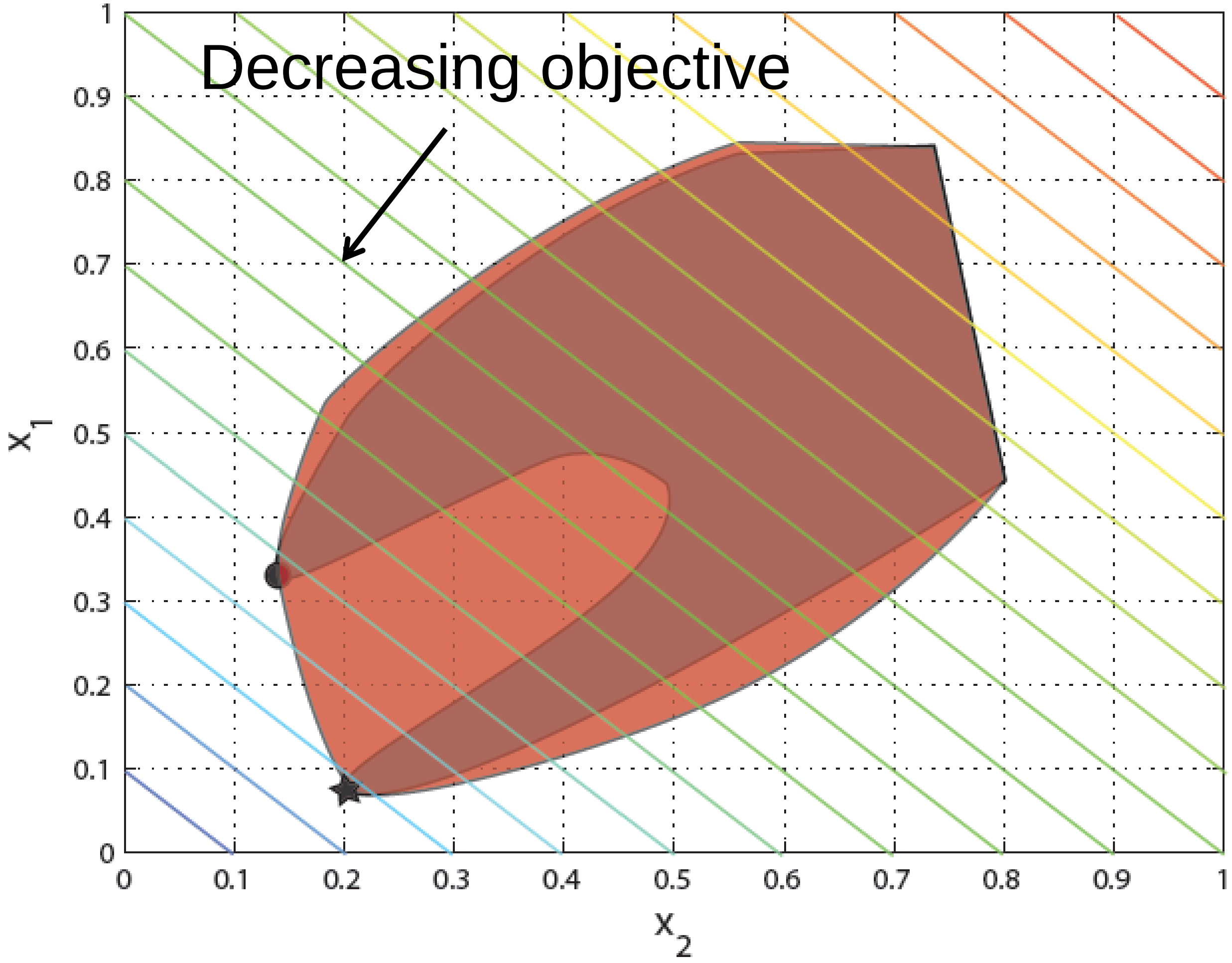
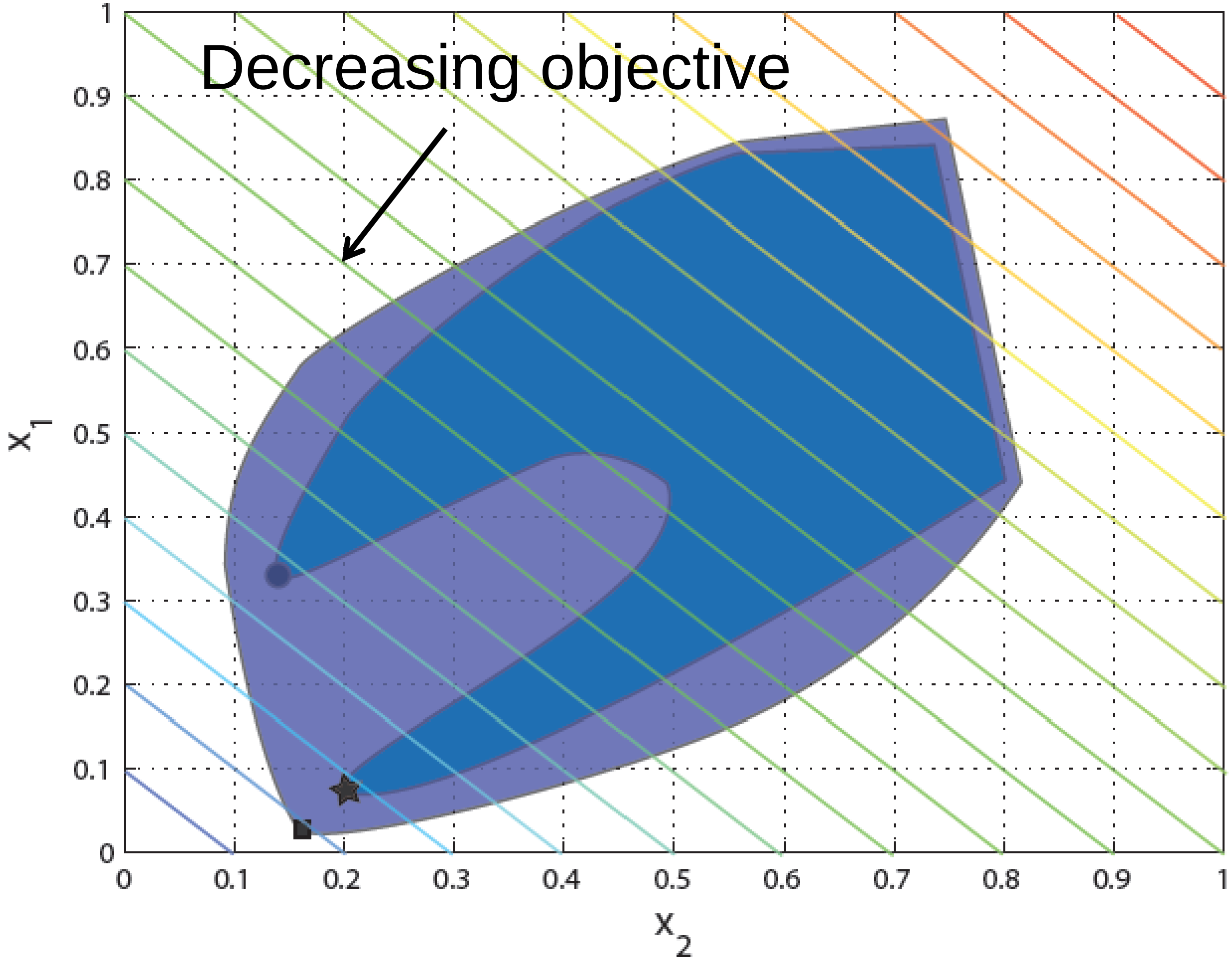


Relaxation finds global optimum

Convex Relaxation



Relaxation does not find global optimum

Semidefinite Programming

- Convex optimization
- Interior point methods solve for the global optimum in polynomial time

$$\min_{\mathbf{W}} \operatorname{trace}(\mathbf{B}\mathbf{W})$$

subject to

$$\operatorname{trace}(\mathbf{A}_i\mathbf{W}) = c_i$$

$$\mathbf{W} \succeq 0$$

where B and $\mathbf{A}_i$ are specified symmetric matrices

Recall: $\operatorname{trace}\left(\mathbf{A}^\mathsf{T}\mathbf{W}\right) = \mathbf{A}_{11}\mathbf{W}_{11} + \mathbf{A}_{12}\mathbf{W}_{12} + \ldots + \mathbf{A}_{nn}\mathbf{W}_{nn}$

$\mathbf{W} \succeq 0$ if and only if $\operatorname{eig}(\mathbf{W}) \geq 0$

Moment Relaxations

Preliminaries

- Exploit moment-based semidefinite relaxations for polynomial optimization problems [Lasserre '01]

$$\min_x f(x) \quad \text{subject to} \quad g_i(x) \geq 0$$

where $f(x)$ and $g_i(x)$ are polynomial functions of $x = \begin{bmatrix} V_{d1} & V_{d2} & \dots & V_{dn} & V_{q1} & V_{q2} & \dots & V_{qn} \end{bmatrix}^\intercal$

- Define linear functional L_y with polynomial argument $h(x)$

$$h(x) = \sum_{\alpha \in \mathbb{N}^{2n}} h_\alpha x^\alpha \qquad L_y\{h(x)\} = \sum_{\alpha \in \mathbb{N}^{2n}} h_\alpha y_\alpha$$

- Define vector x_d containing all monomials up to order d

$$x_d = \begin{bmatrix} 1 & V_{d1} & \dots & V_{qn} & V_{d1}^2 & V_{d1}V_{d2} & \dots & V_{qn}^2 & V_{d1}^3 & V_{d1}^2 V_{d2} & \dots & V_{qn}^d \end{bmatrix}^\intercal$$

Order-d Moment Relaxation

$$\min_x \; f(x) \quad \text{s.t.}$$
$$g_i(x) \geq 0$$

Add redundant constraints

$$\min_x \; f(x) \quad \text{s.t.}$$
$$g_i(x)\, x_{d-1} x_{d-1}^\top \succeq 0$$
$$x_d x_d^\top \succeq 0$$

Convex relaxation via "lifting"

$$\min_y \; L_y\{f(x)\} \quad \text{s.t.}$$
$$L_y\{g_i(x)\, x_{d-1} x_{d-1}^\top\} \succeq 0$$
$$L_y\{x_d x_d^\top\} \succeq 0$$

[Lasserre '01, Parillo '03]

- Increasing d yields a tighter relaxation but has a computational cost
- Recover global optimum if $\text{rank}\left(L_y\{x\, x^\top\}\right) = 1$

Illustrating the Capabilities of the Moment Relaxations

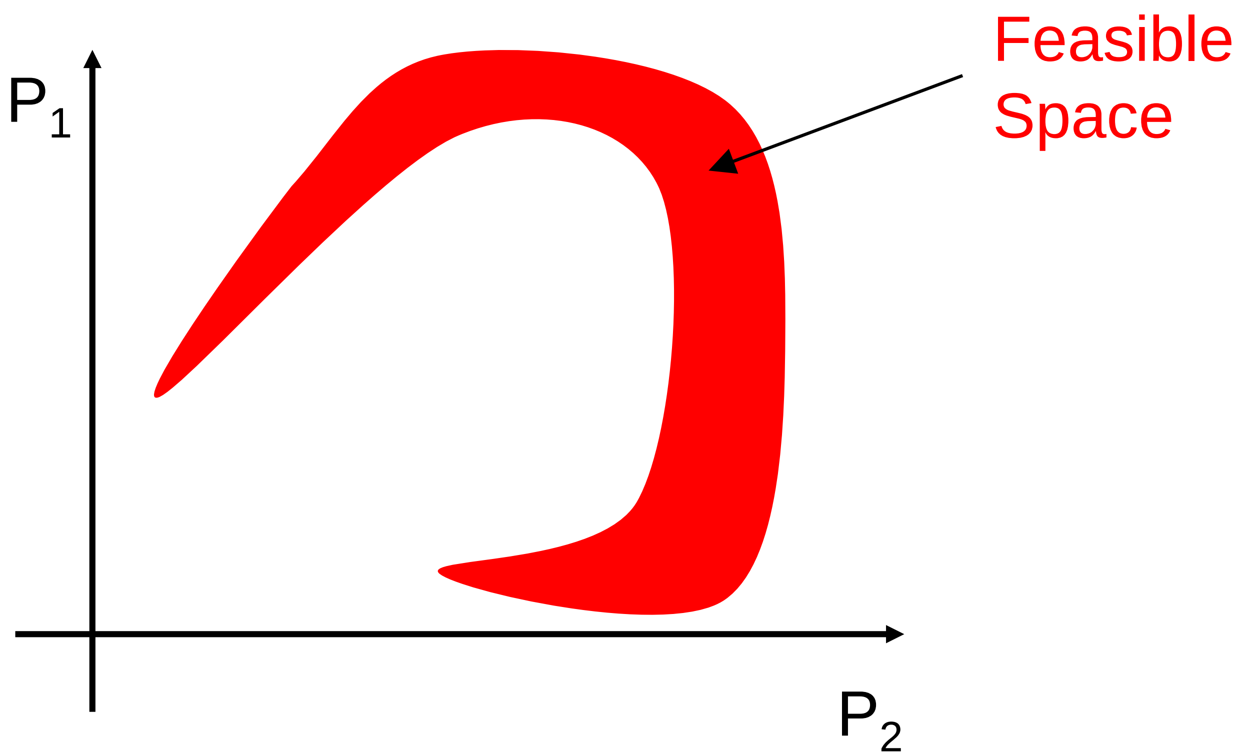
Test Case Results

- Second- and third-order moment-based relaxations globally solve many small OPF problems [M. & Hiskens '14]

Case	Number of Buses	Parameters	Minimum Order
Lesieutre, Molzahn, Borden, & DeMarco '11	3	50 MVA line limit	2
Molzahn, Lesieutre, & DeMarco '14	3	100 MVA line limit	2
Bukhsh, Grothey, McKinnon & Trodden '13	3	$P_{D3} = 17.17$ per unit	2
Lesieutre & Hiskens '05	5		2
Bukhsh, Grothey, McKinnon & Trodden '13	5	$-50 \leq Q_5^{\min} \leq -27.36$ MVAR $-27.35 \leq Q_5^{\min} \leq -27.04$ MVAR $-27.03 \leq Q_2^{\min} \leq 0$ MVAR	2 3 2
Bukhsh, Grothey, McKinnon & Trodden '13	9		2

We were perplexed!

- The unconstrained minimum could be at an infeasible point inside the convex hull of the constraints.
  - How does the second-order relaxation find the global solution?



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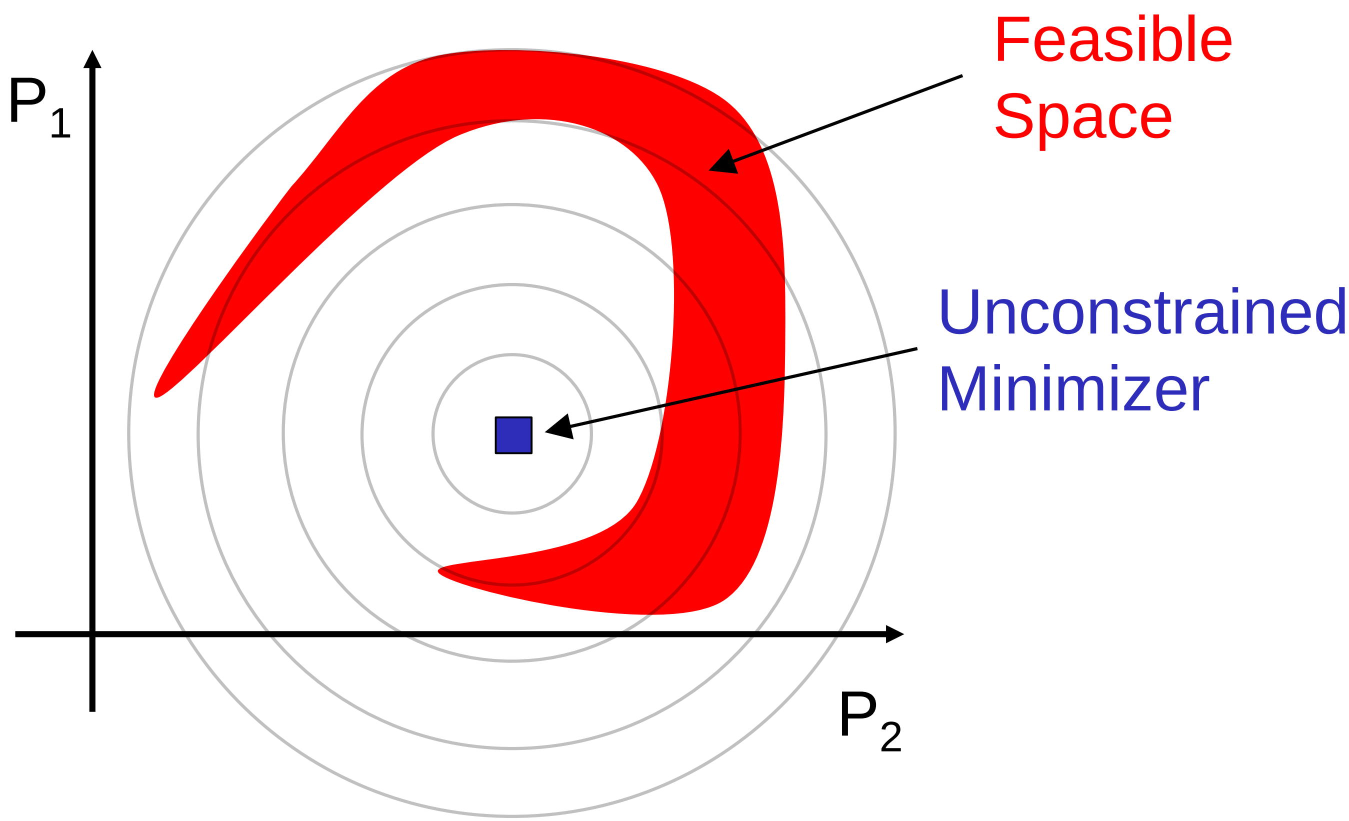

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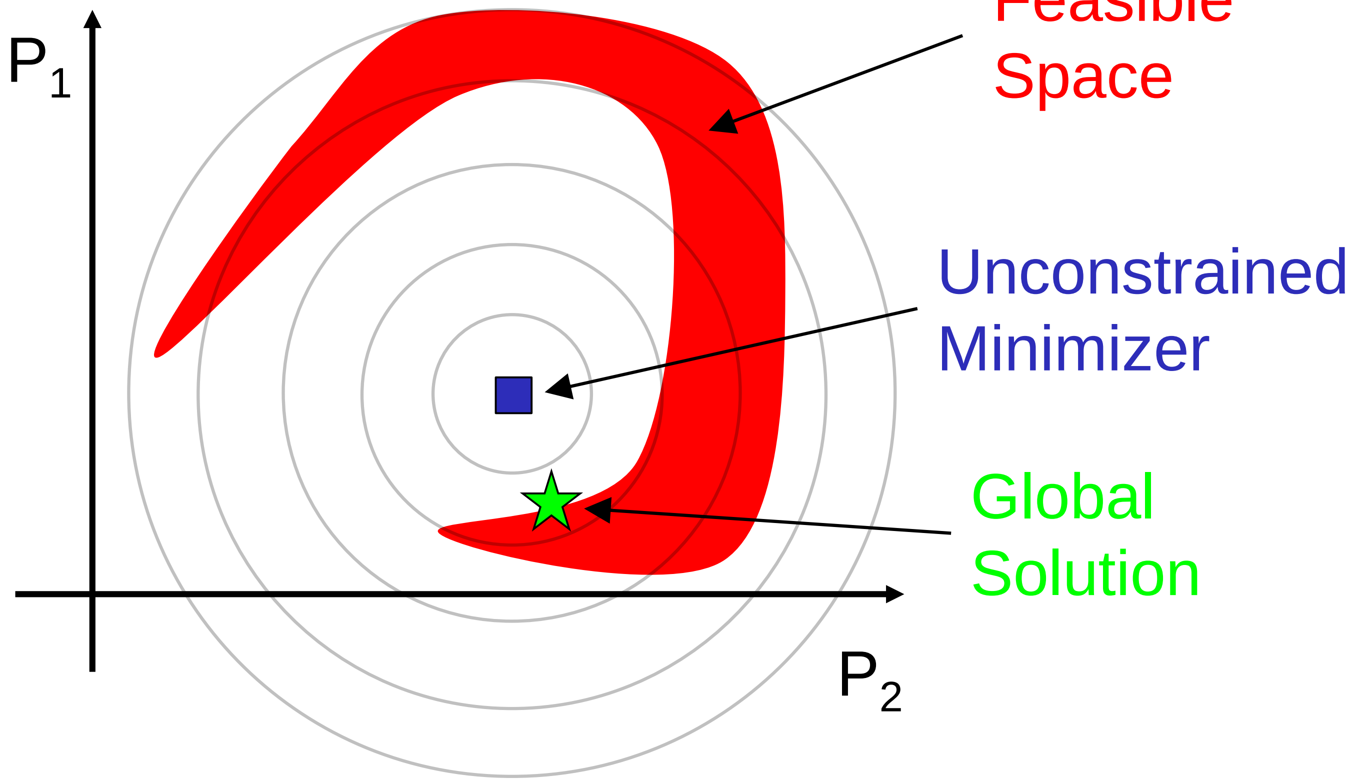

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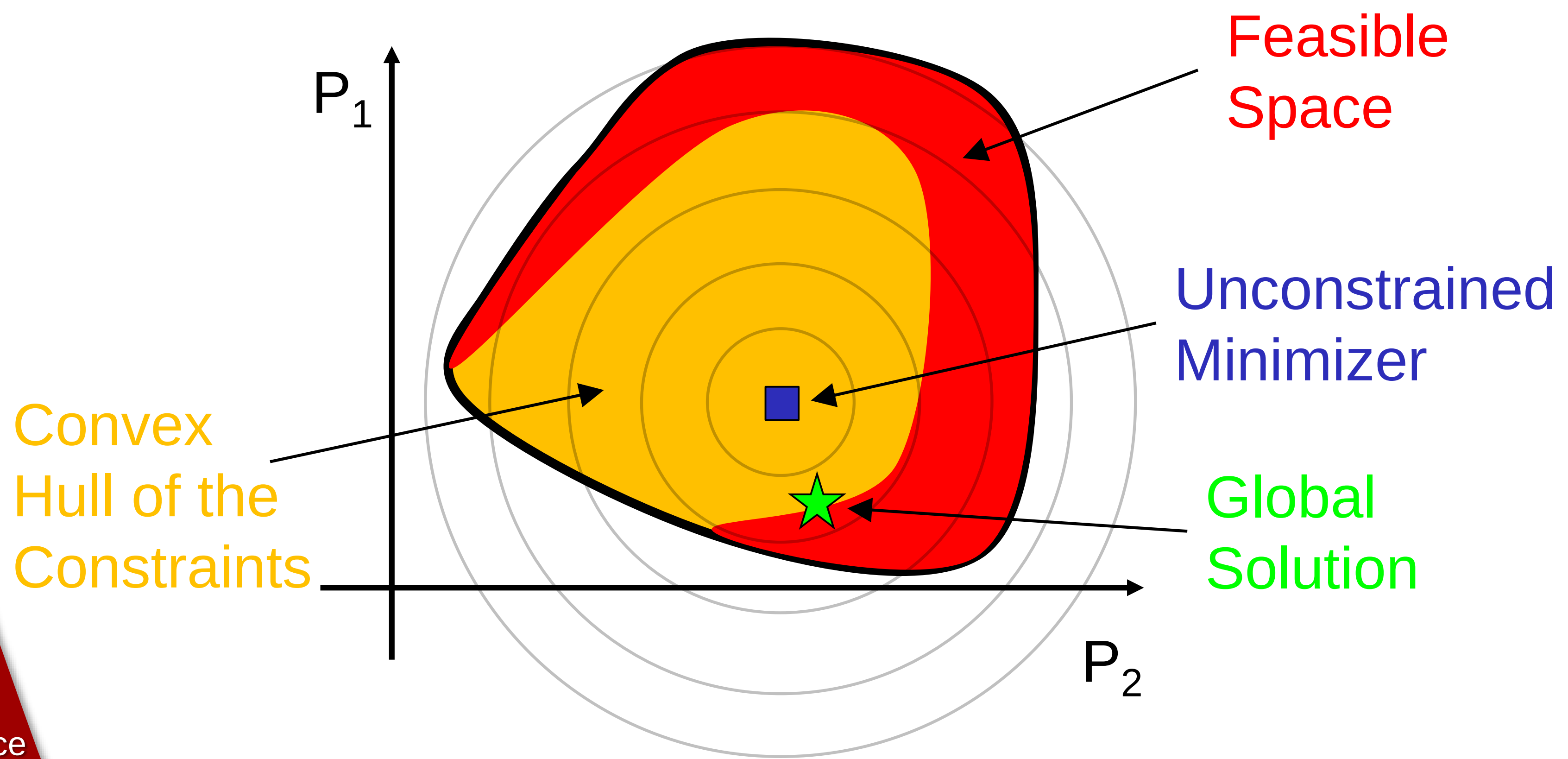


Explanation: The Lifted Cost Function

- Quadratic cost of active power generation is quartic in voltage components:

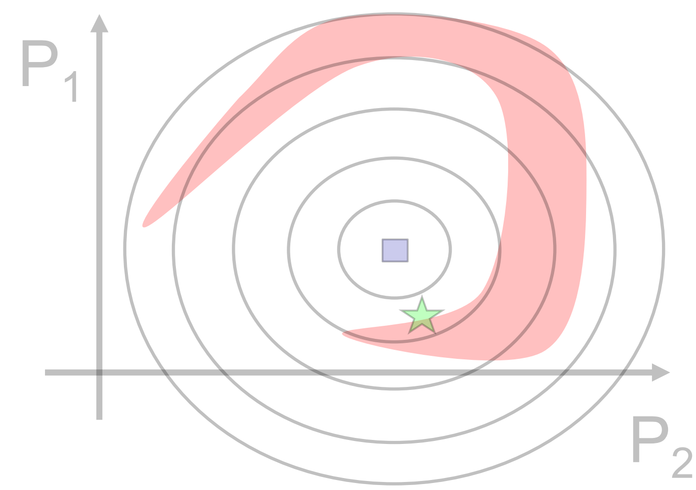
$$\min_{y} \; L_y \left\{ \sum_{i \in \mathcal{G}} c_{2,i} \left(P_{gi} \left(x \right) \right)^2 + c_{1,i} P_{gi} \left(x \right) + c_{0,i} \right\}$$

$$\text{s.t.} \quad L_y \left\{ g_i \left(x \right) x_1 \, x_1^\intercal \right\} \succeq 0$$

$$L_y \left\{ x_2 \, x_2^\intercal \right\} \succeq 0$$

- Infeasible points in the non-convex problem have high-cost in the relaxation

Non-Convex Problem



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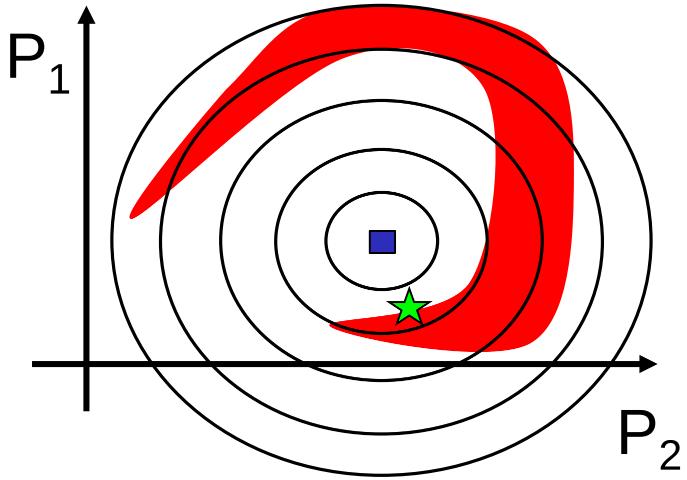

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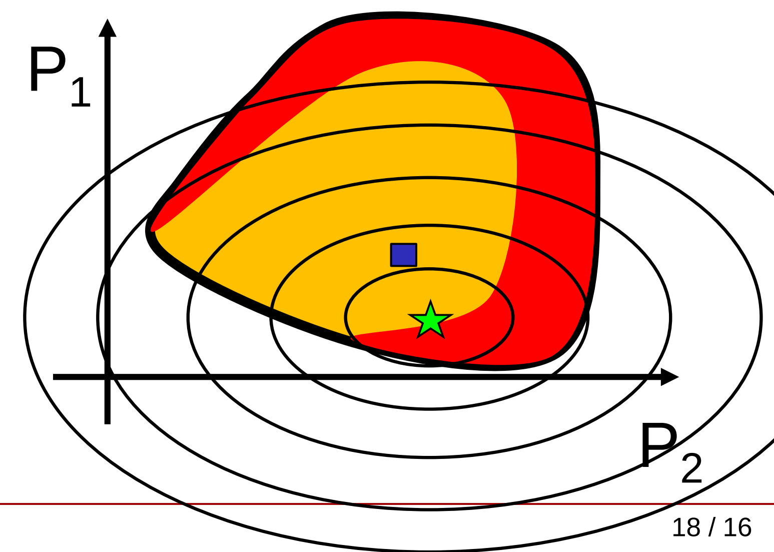
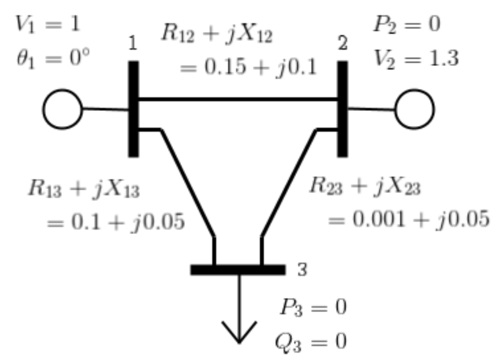
Second-Order Moment Relaxation

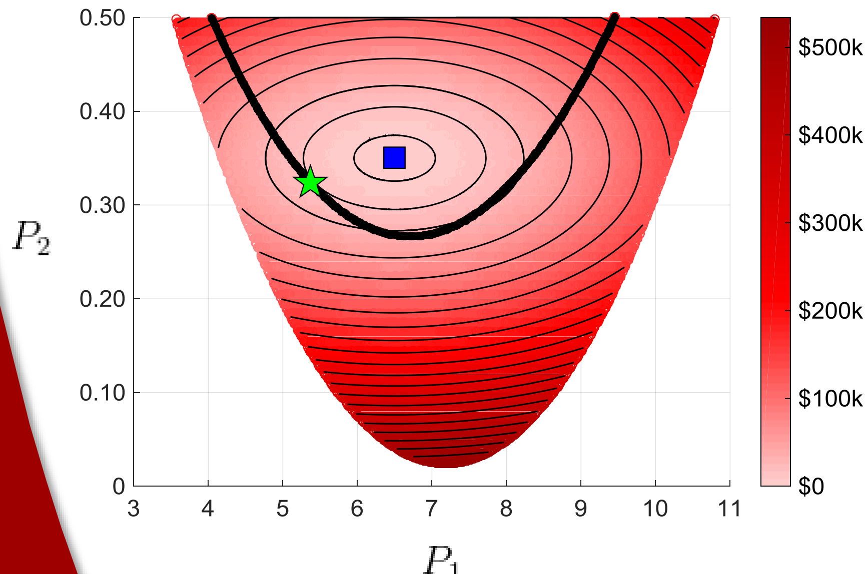


Hole in the Feasible Space

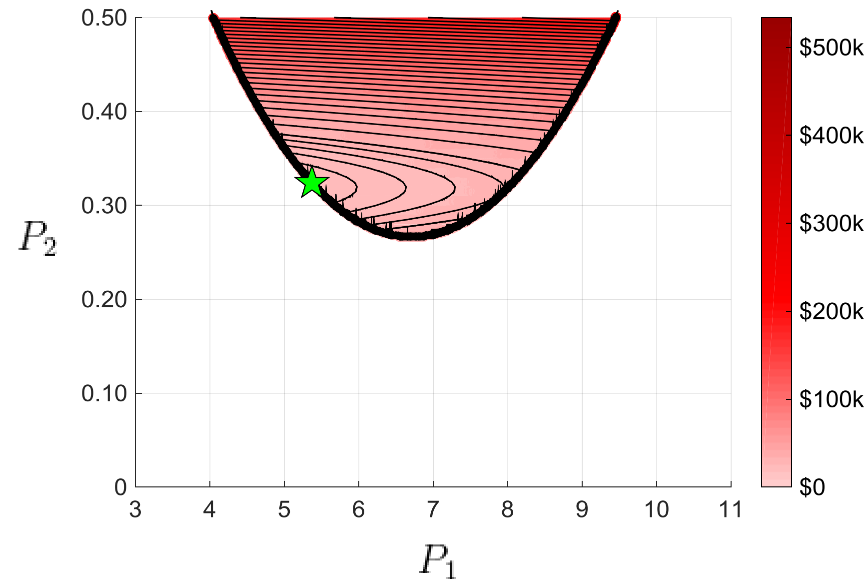
- Three-bus example OPF problem
  [Molzahn, Baghsorkhi, & Hiskens '15]



First-Order Relaxation

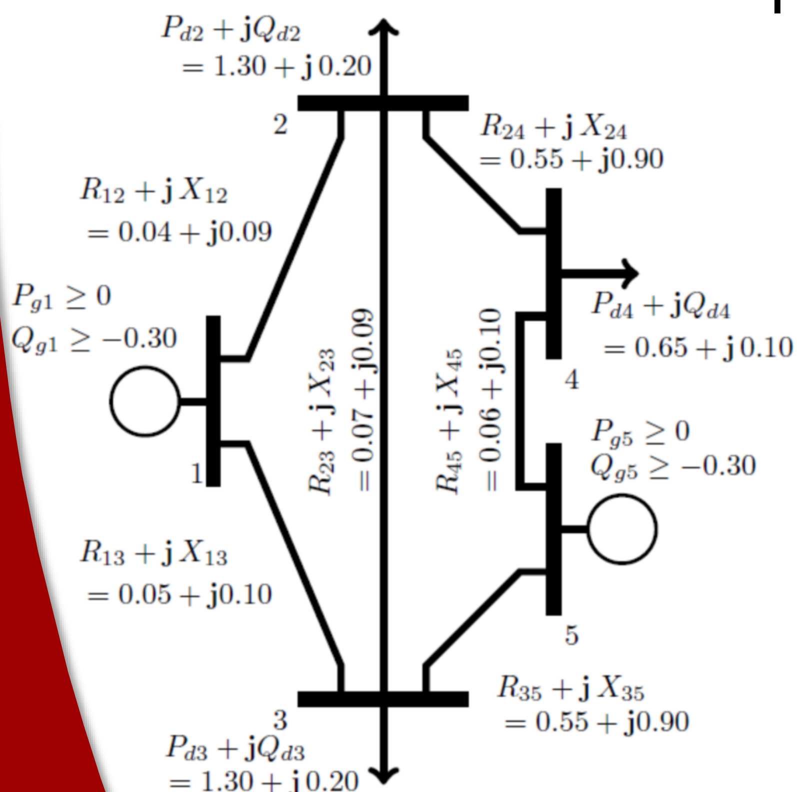


Second-Order Relaxation

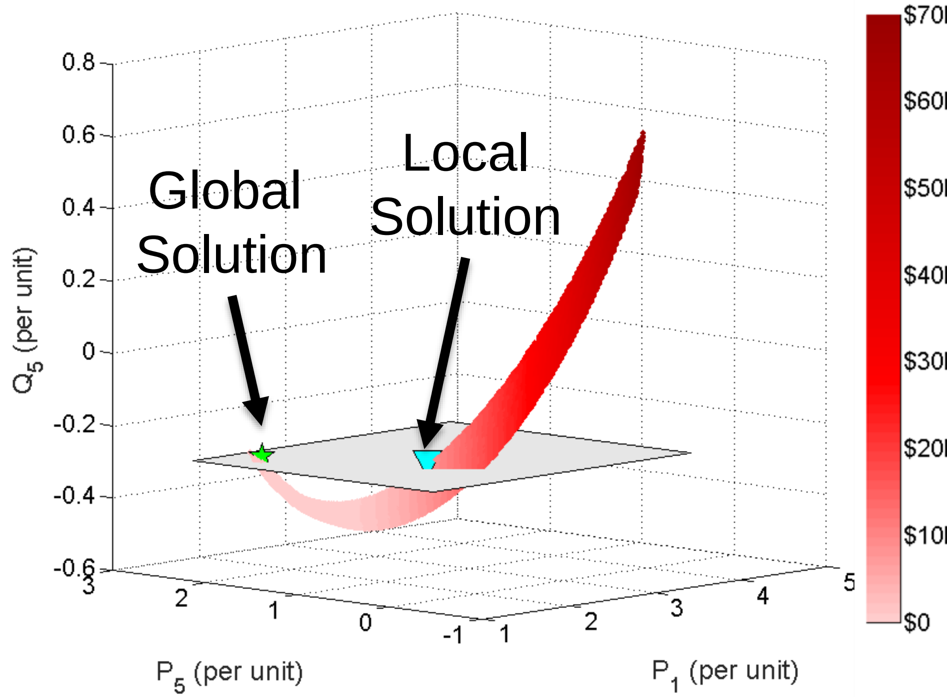


Disconnected Feasible Space

- Five-bus example OPF problem (modified objective) [Bukhsh et al. '13]



Feasible Space of OPF Problem



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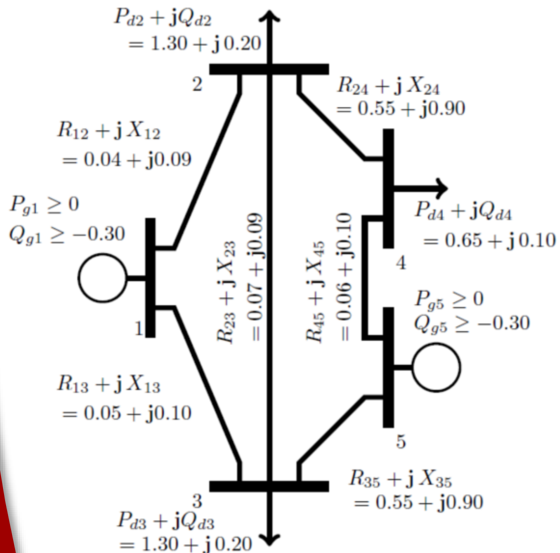
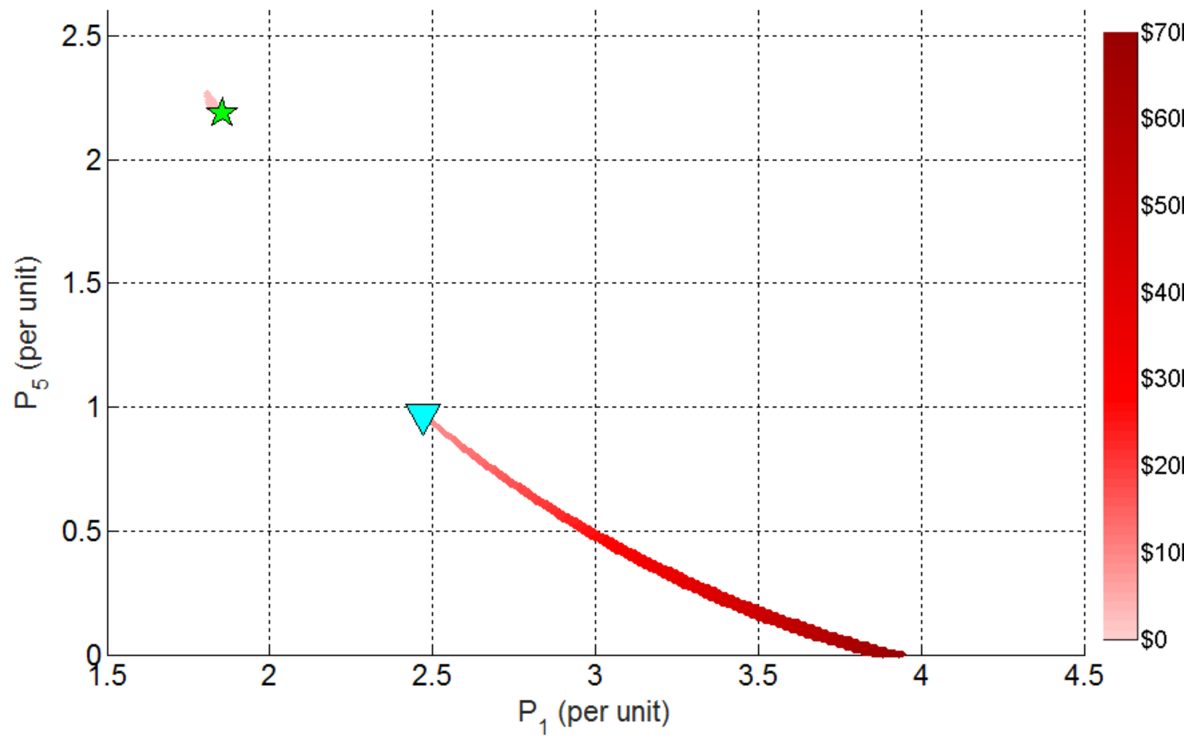


Feasible Space of OPF Problem



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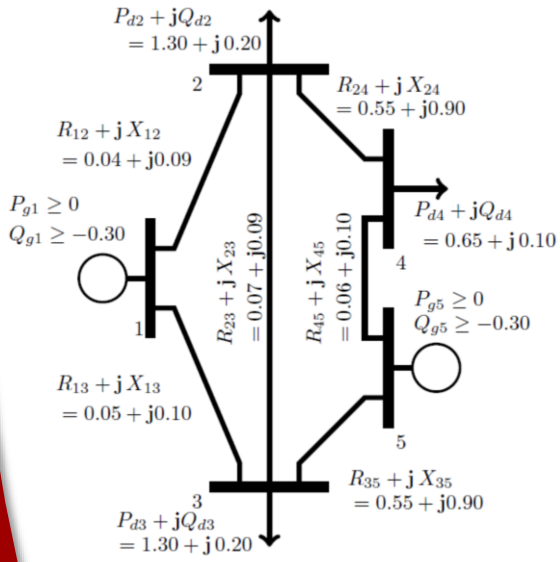

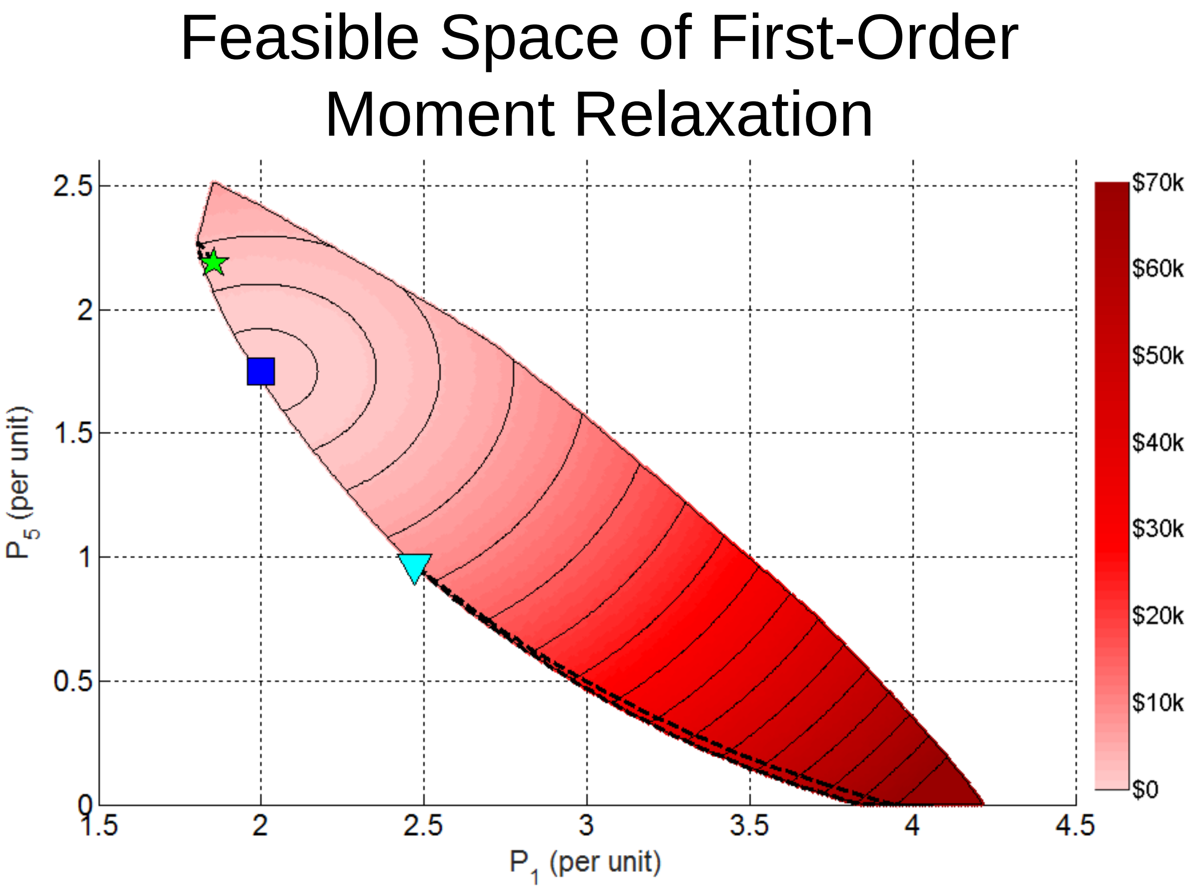

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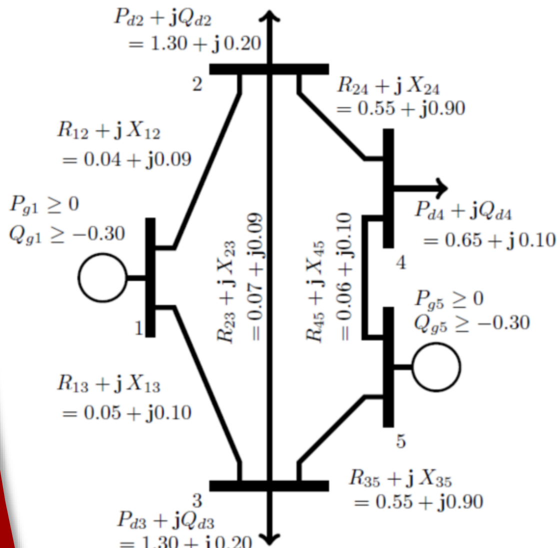
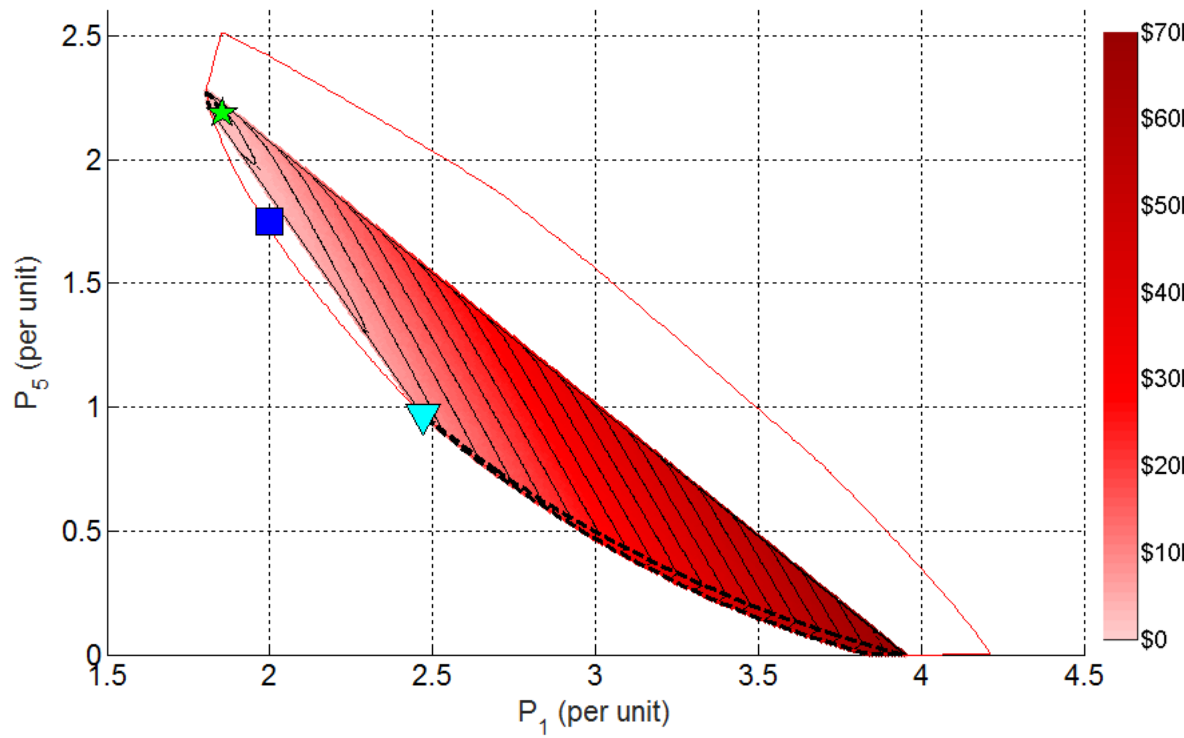


Feasible Space of Second-Order Moment Relaxation



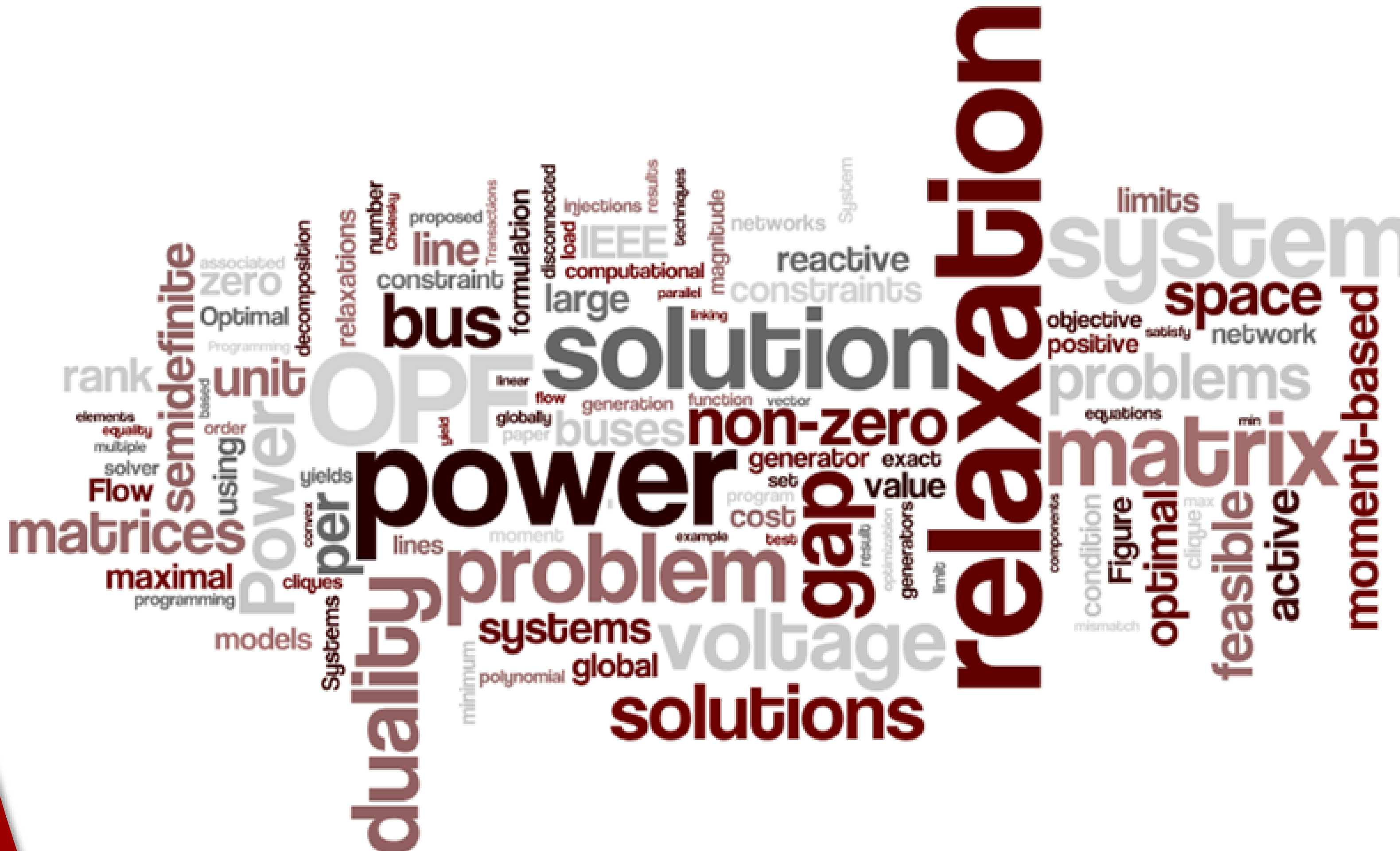
Second-order moment relaxation finds the global solution

Conclusion

Conclusion

- Moment relaxations find global solutions to many OPF problems
- Illustration of the moment relaxations' capabilities to find global optima despite infeasible points inside the constraints' convex hull

Questions?


relaxation
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matrix
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duality
Power
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space
moment-based
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networks
magnitude
Figure
condition
using
yields
lines
moment
polynomial
Systems
order
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minimum
set
program
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equations

The Cost Function

- First-order relaxation:

$$\min_{y} \; L_y\left\{\sum_{i\in\mathcal{G}} \alpha_i\right\}$$

$$\text{s.t.} \quad \alpha_i \geq L_y\left\{c_{2,i}\left(P_{gi}\left(x\right)\right)^2 + c_{1,i}P_{gi}\left(x\right) + c_{0,i}\right\}$$

$$L_y\left\{g_i\left(x\right)\right\} \succeq 0 \qquad L_y\left\{x_1\, x_1^{\intercal}\right\} \succeq 0$$

Schur Complement Reformulation as an SOCP

- Second-order relaxation:

$$\min_{y} \; L_y\left\{\sum_{i\in\mathcal{G}} c_{2,i}\left(P_{gi}\left(x\right)\right)^2 + c_{1,i}P_{gi}\left(x\right) + c_{0,i}\right\}$$

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