Robust Beamforming based on Minimum Dispersion Criterion

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Introduction

What is Beamformer?

Beamformer is a spatial filter.

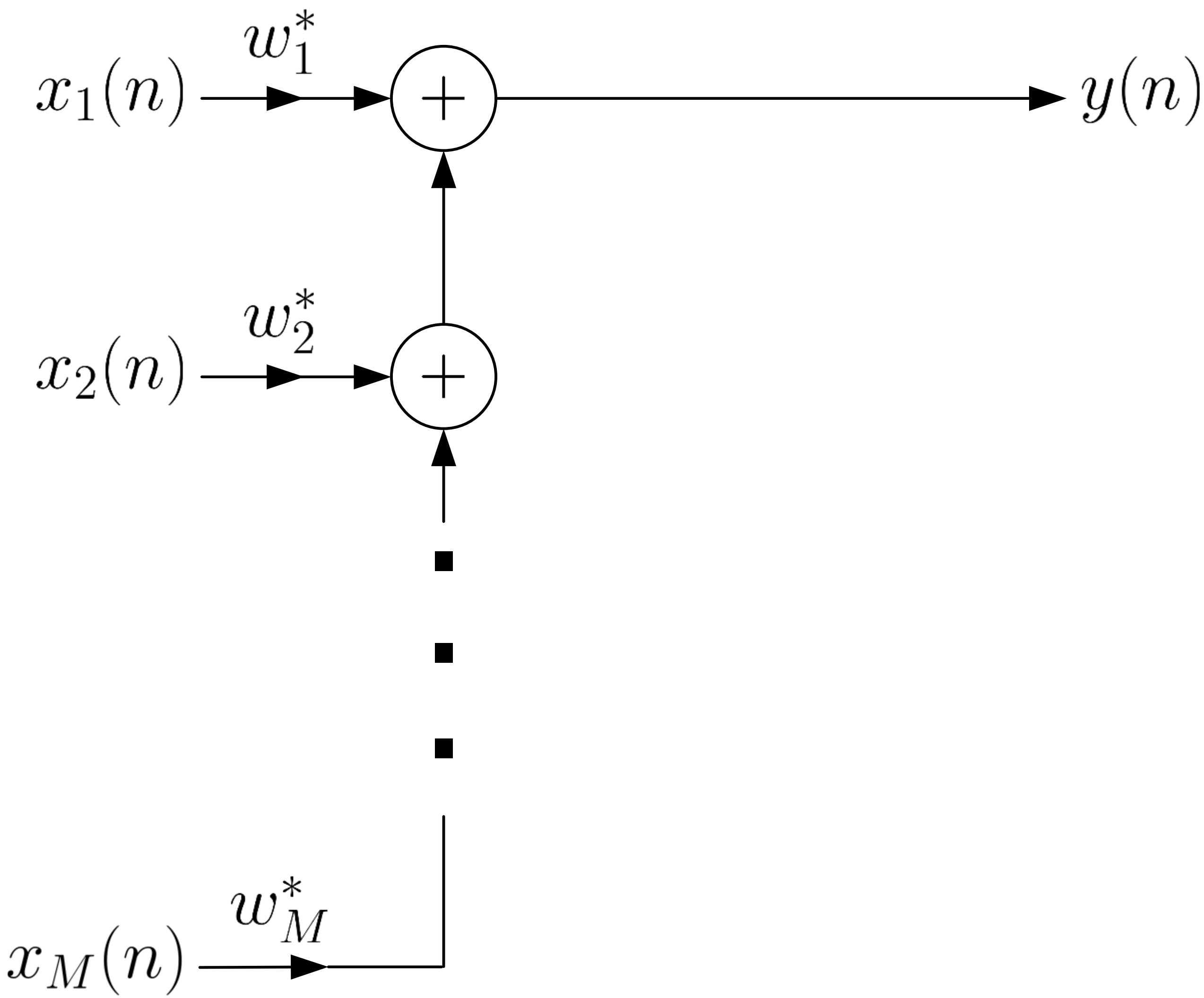
Analogous to the time-domain filter, e.g., tapped delay line

$$y(n) = \sum_{m=1}^{M} w_m^* x(n-m)$$

The beamformer output is:

$$y(n) = \sum_{m=1}^{M} w_m^* x_m(n)$$

where $x_m(n)$ is signal received at the mth spatially separated sensor at time n and w_m is the corresponding weight.



Typical Beamformer

Why Beamforming?

By properly choosing $\{w_m\}$, signal-of-interest (SOI) at a possibly known direction is enhanced while surrounding interferences and noise at other directions are suppressed.

Application areas include:

- Radar
- Sonar
- Communications
- Audio and Speech Processing
- Astronomy
- Seismology
- Biomedicine
- Brain-Computer Interaction
- Assisted Living

Approaches for Beamforming

In data-independent approach, the weights do not depend on the array data and are computed to provide a specified response for all SOI and/or interference scenarios.

Data-dependent approach determines the weights as a function of the received signals according to an optimization criterion, and is able to provide higher resolution and interference rejection capability.

The standard optimization criterion is to maximize signal-to-interference-plus-noise ratio (SINR).

Foundation of Data-Dependent Beamformer Design

Let $\boldsymbol{x}(n) = [x_1(n) \ \cdots \ x_M(n)]^T \in \mathbb{C}^M$ and consider the narrowband case. The array output can be modeled as:

$$\boldsymbol{x}(n) = s(n)\boldsymbol{a} + \boldsymbol{i}(n) + \boldsymbol{v}(n), \quad \boldsymbol{i}(n) = \sum_{i=1}^{I} s_i(n)\boldsymbol{a}_i$$

- $s(n) \in \mathbb{C}$ is SOI with steering vector $\boldsymbol{a} \in \mathbb{C}^M$
- $\{s_i(n)\}_{i=1}^I$ are I interferences with steering vectors $\{\boldsymbol{a}_i\}_{i=1}^I$
- $\boldsymbol{v}(n) \in \mathbb{C}^M$ is additive noise vector
- SOI, interferences and noise are zero-mean and independent of each other

The output of the beamformer is:

$$y(n) = \boldsymbol{w}^H \boldsymbol{x}(n), \quad \boldsymbol{w} = [w_1 \ \cdots \ w_M]^T$$

The SINR is defined as:

$$\mathrm{SINR}=\frac{\mathbb{E}\{|s(n)\boldsymbol{w}^H\boldsymbol{a}|^2\}}{\mathbb{E}\{|\boldsymbol{w}^H(\boldsymbol{i}(n)+\boldsymbol{v}(n))|^2\}}=\frac{\sigma_s^2|\boldsymbol{w}^H\boldsymbol{a}|^2}{\boldsymbol{w}^H\boldsymbol{R}_{i+v}\boldsymbol{w}}$$

where $\sigma_s^2=\mathbb{E}\{|s(n)|^2\}$ is the SOI power and $\boldsymbol{R}_{i+v}$ is the interference-plus-noise covariance matrix.

To maximize the SINR, the standard strategy for obtaining a unique solution is to minimize $\boldsymbol{w}^H\boldsymbol{R}_{i+v}\boldsymbol{w}$ subject to the linear constraint $\boldsymbol{w}^H\boldsymbol{a}=1$, which results in minimum variance distortionless response (MVDR) or Capon beamformer formulation:

$$\min_{\boldsymbol{w}}\ \left(\mathrm{E}\{|y(n)|^2\}=\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}\right),\quad \text{s.t. } \boldsymbol{a}^H\boldsymbol{w}=1,\quad \boldsymbol{R}=\mathbb{E}\{\boldsymbol{x}(n)\boldsymbol{x}^H(n)\}$$

because

$$\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}=\boldsymbol{w}^H\boldsymbol{R}_{i+n}\boldsymbol{w}+\sigma_s^2\left|\boldsymbol{w}^H\boldsymbol{a}\right|^2$$

The solution is:

$$\boldsymbol{w}_{\text{MVDR}} = \frac{\boldsymbol{R}^{-1}\boldsymbol{a}}{\boldsymbol{a}^H\boldsymbol{R}^{-1}\boldsymbol{a}}$$

In practice, $\boldsymbol{R}$ is substituted by its estimate based on finite number of samples, say, $\boldsymbol{X} = [\boldsymbol{x}(1) \ \cdots \ \boldsymbol{x}(N)]$:

$$\hat{\boldsymbol{R}} = \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{x}(n)\boldsymbol{x}^H(n) = \frac{1}{N}\boldsymbol{X}\boldsymbol{X}^H$$

Even in the ideal scenarios, performance degradation is expected particularly when N is small.

Performance deterioration in real environments where there also exists model mismatch including errors in SOI's direction and sensor positions, imperfect array calibration, and signal waveform distortion.

All these uncertainties can be absorbed in the mismatch of the steering vector $\boldsymbol{a}$.

To achieve robustness against mismatch in SOI's direction, linearly constrained minimum variance (LCMV) beamformer generalizes MVDR to multiple linear constraints:

$$\begin{aligned}\boldsymbol{w}_{\mathrm{LCMV}} &= \arg\min_{\boldsymbol{w}} \ \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w}, \quad \text{s.t. } \Phi^H \boldsymbol{w} = \boldsymbol{g} \\ &= \boldsymbol{R}^{-1} \Phi \left(\Phi^H \boldsymbol{R}^{-1} \Phi \right)^{-1} \boldsymbol{g}\end{aligned}$$

where $\Phi = [\boldsymbol{\phi}_1 \ \cdots \ \boldsymbol{\phi}_L] \in \mathbb{C}^{M \times L}$ contains L steering vectors corresponding to a small spread of angles around the nominal direction-of-arrival (DOA) and $\boldsymbol{g} = [g_1 \ \cdots \ g_L]^T$ is usually assigned with all elements being one.

To achieve robustness against arbitrary steering vector mismatch, the observed steering vector is modeled as

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{e}$$

where $\boldsymbol{e} \in \mathbb{C}^M$ is steering error vector lying in an uncertainty set. A conventional choice of uncertainty region is a sphere $\mathcal{E} = \{\boldsymbol{e} \mid \|\boldsymbol{e}\| \le \varepsilon\}$ with ε being the radius, leading to:

$$|(\boldsymbol{a}+\boldsymbol{e})^H \boldsymbol{w}| \ge 1, \text{ for all } \boldsymbol{e} \in \mathcal{E} \Rightarrow \mathrm{Re}(\boldsymbol{a}^H \boldsymbol{w}) \ge \varepsilon \|\boldsymbol{w}\| + 1$$

This is worst-case performance optimization approach:

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w}, \quad \text{s.t. } \mathrm{Re}(\boldsymbol{a}^H \boldsymbol{w}) \ge \varepsilon \|\boldsymbol{w}\| + 1$$

which is a nonconvex optimization problem with infinitely many quadratic constraints. The uncertainty region can be generalized to an ellipsoid. Both problems can be converted to a second-order cone program (SOCP).

Beamforming with Minimum Dispersion Criterion

Minimum variance criterion is conventionally used where

$$\mathrm{E}\{|y(n)|^2\} = \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} \qquad \text{or} \qquad \boldsymbol{w}^H \boldsymbol{X} \boldsymbol{X}^H \boldsymbol{w} = |\boldsymbol{X}^H \boldsymbol{w}|_2^2$$

is minimized.

The ℓ_2-norm is optimum for Gaussian data but not for non-Gaussian signals.

To handle non-Gaussian data, we propose minimum dispersion criterion where the cost function to be minimized:

$$\|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad p > 0$$

which generalizes the minimum variance criterion.

For $p \in (2, \infty]$, higher-order statistics are exploited, which is expected to outperform that of $p = 2$ for sub-Gaussian data.

Common sub-Gaussian data include PSK, QAM, QPSK, sonar, and GPS navigation signals.

For $p \in (0, 2)$, lower-order statistics are exploited, and we expect superiority over that of $p = 2$ for super-Gaussian data.

Common super-Gaussian data include speech, biomedical signals and radar clutter, which can be impulsive.

Minimum dispersion based beamforming is summarized as:

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad \text{s.t. } \boldsymbol{w} \in \mathcal{S}, \quad \mathcal{S} = \mathcal{S}_{\text{linear}} \text{ or } \mathcal{S} = \mathcal{S}_{\text{nonlinear}}$$

where $\mathcal{S}_{\text{linear}}$ and $\mathcal{S}_{\text{nonlinear}}$ are linear and nonlinear constraints.

This formulation in fact generalizes many existing beamformers:

When $p = 2$ and $\mathcal{S}_{\mathrm{linear}}$ corresponds to $\boldsymbol{w}^H \boldsymbol{a} = 1$ or $\boldsymbol{\Phi}^H \boldsymbol{w} = \boldsymbol{g}$, it becomes the MVDR or LCMV beamformer.

When $p = 2$ and $\mathcal{S}_{\mathrm{nonlinear}}$ corresponds to the uncertain region modeled as a sphere or ellipsoid, it becomes the beamformer based on worst-case performance optimization.

When we use $p > 2$ and $p < 2$ in the objective function, higher SINR will be yielded for sub-Gaussian and super-Gaussian signals, respectively.

Algorithm Design Examples

1. Minimum Dispersion Distortionless Response (MDDR) [1]

The beamformer formulation is:

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad \text{s.t. } \boldsymbol{w}^H \boldsymbol{a} = 1$$

Here we focus on $p > 1$ so that $\|\boldsymbol{X}^H \boldsymbol{w}\|_p^p$ is convex.

Denoting $\boldsymbol{y} = [y(1) \ \cdots \ y(N)]^T$, we have:

$$\|\boldsymbol{X}^H \boldsymbol{w}\|_p^p = \|\boldsymbol{y}^*\|_p^p = \sum_{n=1}^{N} |y(n)|^p = \sum_{n=1}^{N} |y(n)|^{p-2} |y(n)|^2 = \|\boldsymbol{\Delta} \boldsymbol{y}^*\|_2^2$$

where

$$\boldsymbol{\Delta} = \operatorname{diag}\left\{ |y(1)|^{(p-2)/2}, \cdots, |y(N)|^{(p-2)/2} \right\}$$

Further reorganization yields:

$$\|\boldsymbol{X}^H\boldsymbol{w}\|_p^p = \|\Delta\boldsymbol{y}^*\|^2 = \boldsymbol{y}^T\Delta^H\Delta\boldsymbol{y}^* = \boldsymbol{y}^T\boldsymbol{D}(\boldsymbol{w})\boldsymbol{y}^* = \boldsymbol{w}^H\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w})\boldsymbol{X}^H\boldsymbol{w}$$

where

$$\boldsymbol{D}(\boldsymbol{w}) = \mathrm{diag}\left\{|y(1)|^{p-2}, \cdots, |y(N)|^{p-2}\right\}$$

Hence the MDDR beamformer is rewritten as:

$$\min_{\boldsymbol{w}} \ \boldsymbol{w}^H\left(\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w})\boldsymbol{X}^H\right)\boldsymbol{w} \quad \text{s.t. } \boldsymbol{a}^H\boldsymbol{w} = 1$$

which can be solved in an iteratively reweighted manner:

$$\boldsymbol{w}^{k+1} = \frac{\left(\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w}^k)\boldsymbol{X}^H\right)^{-1}\boldsymbol{a}}{\boldsymbol{a}^H\left(\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w}^k)\boldsymbol{X}^H\right)^{-1}\boldsymbol{a}}$$

A more advanced technique for MDDR beamformer is to use complex-valued Newton method with equality constraint [1].

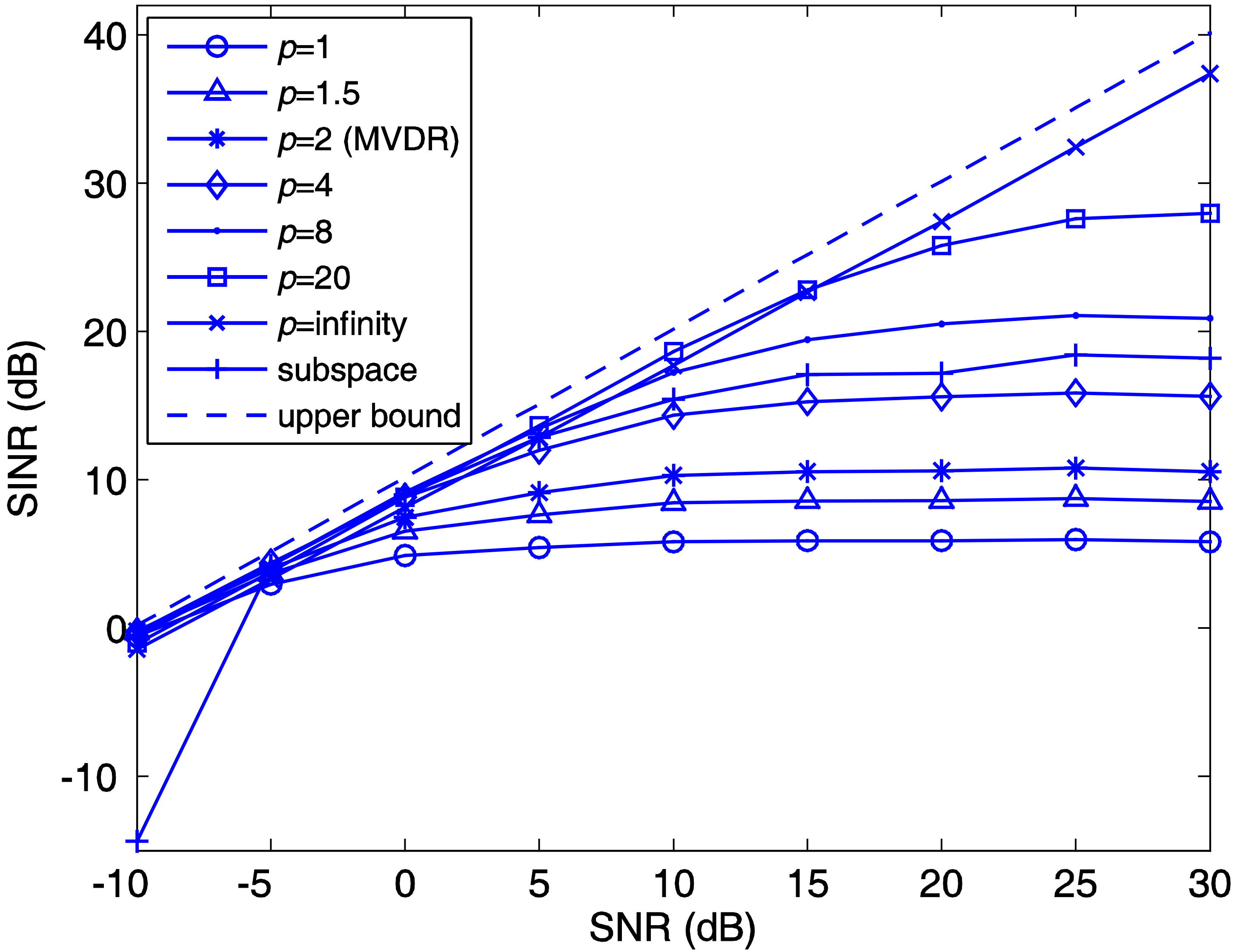
At $p=\infty$, we have:

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H\boldsymbol{w}\|_\infty, \quad \text{s.t. } \boldsymbol{w}^H\boldsymbol{a}=1$$

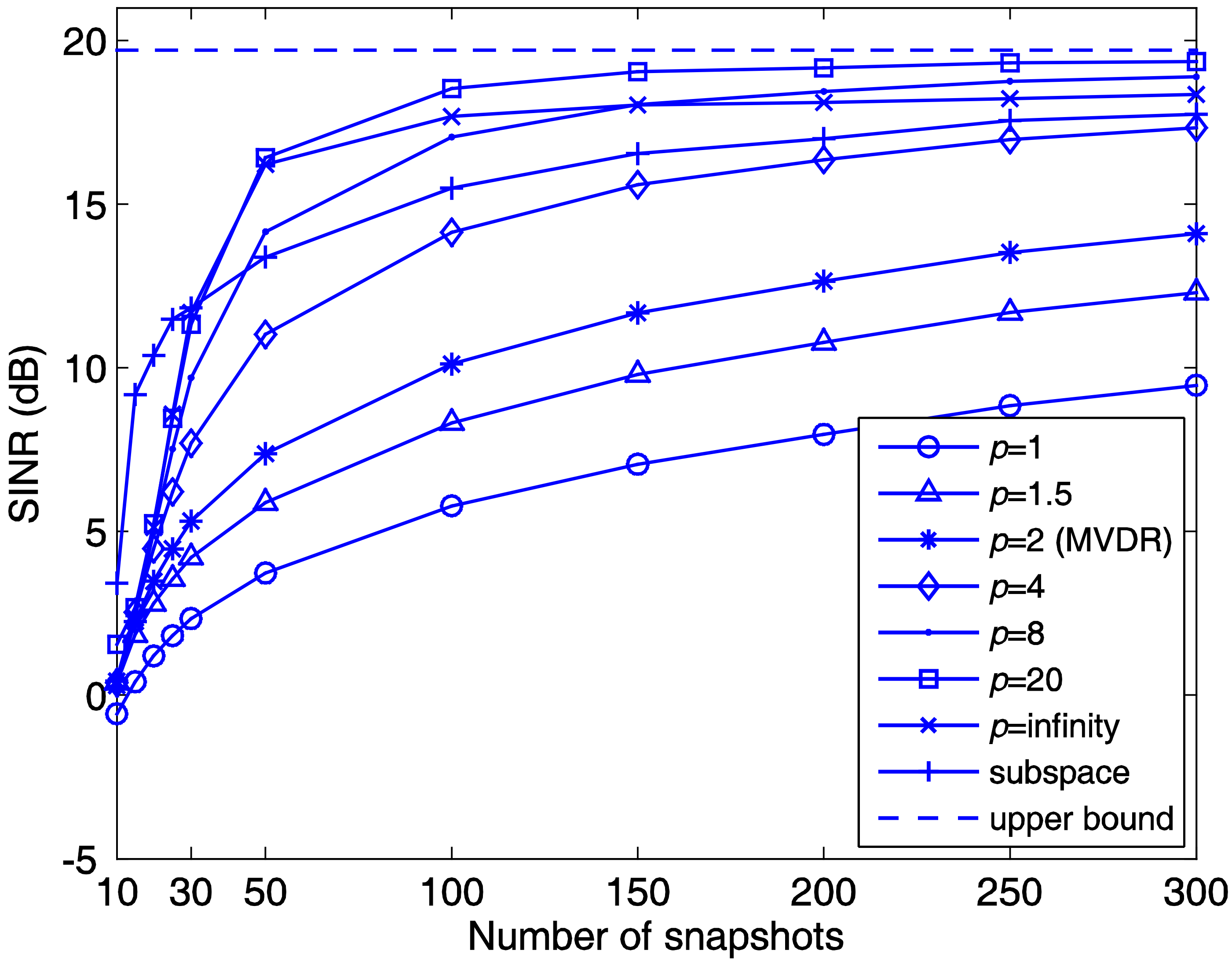
which can be reformulated as SOCP:

$$\begin{aligned}
&\min_{\boldsymbol{w}_R,\boldsymbol{w}_I,\boldsymbol{y}_R,\boldsymbol{y}_I,u} \quad u \\
&\text{s.t. } \sqrt{y_R^2(n)+y_I^2(n)} \leq u, \; n=1,\cdots,N \\
&\begin{bmatrix} \boldsymbol{X}_R & \boldsymbol{X}_I \\ \boldsymbol{X}_I & -\boldsymbol{X}_R \end{bmatrix}^T \begin{bmatrix} \boldsymbol{w}_R \\ \boldsymbol{w}_I \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_R \\ \boldsymbol{y}_I \end{bmatrix} \\
&\begin{bmatrix} \boldsymbol{a}_R & -\boldsymbol{a}_I \\ \boldsymbol{a}_I & \boldsymbol{a}_R \end{bmatrix}^T \begin{bmatrix} \boldsymbol{w}_R \\ \boldsymbol{w}_I \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

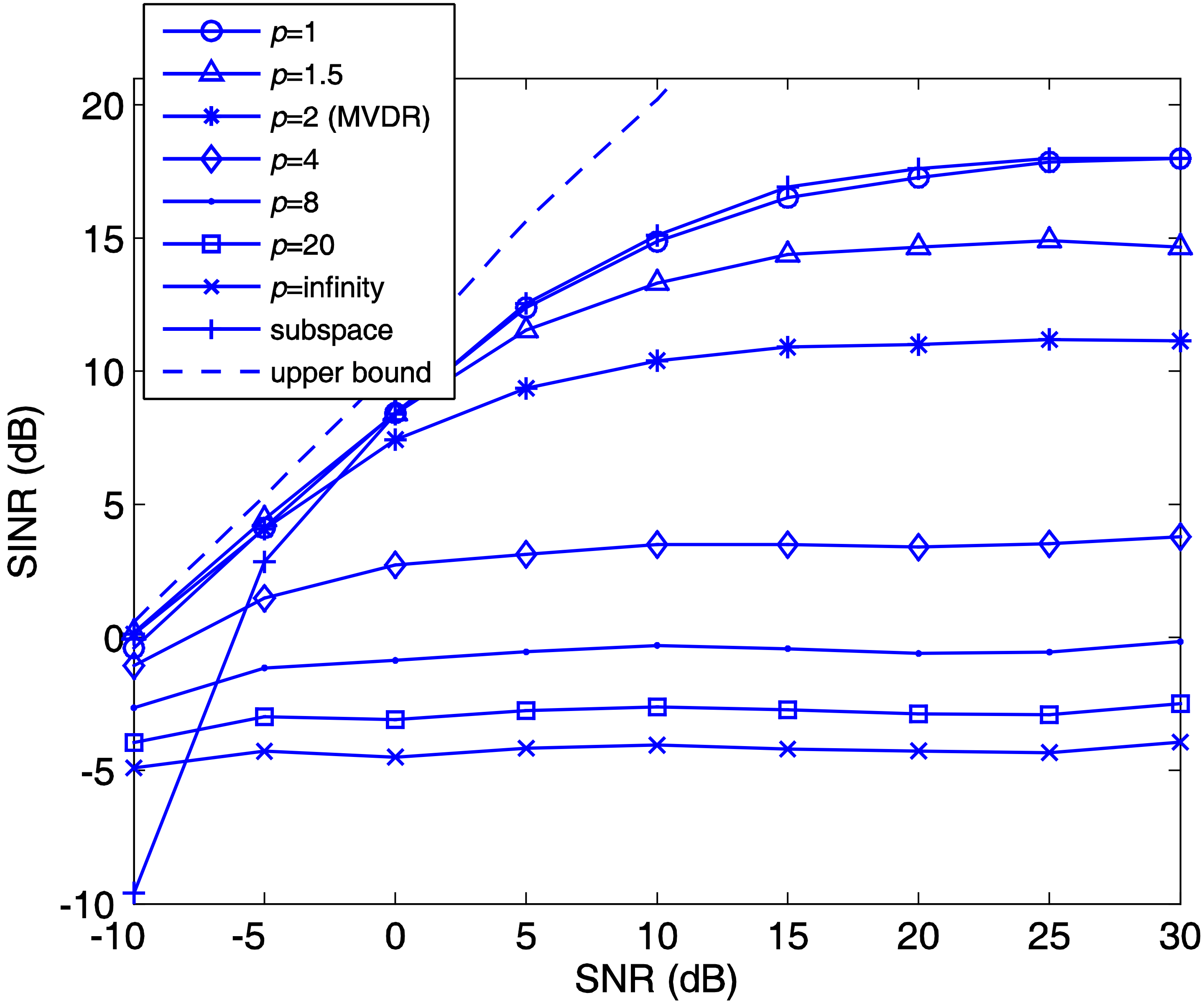
where $\boldsymbol{w}=\boldsymbol{w}_R+\mathrm{j}\boldsymbol{w}_I$, $\boldsymbol{a}=\boldsymbol{a}_R+\mathrm{j}\boldsymbol{a}_I$, $\boldsymbol{y}=\boldsymbol{y}_R+\mathrm{j}\boldsymbol{y}_I$ and $\boldsymbol{X}=\boldsymbol{X}_R+\mathrm{j}\boldsymbol{X}_I$.



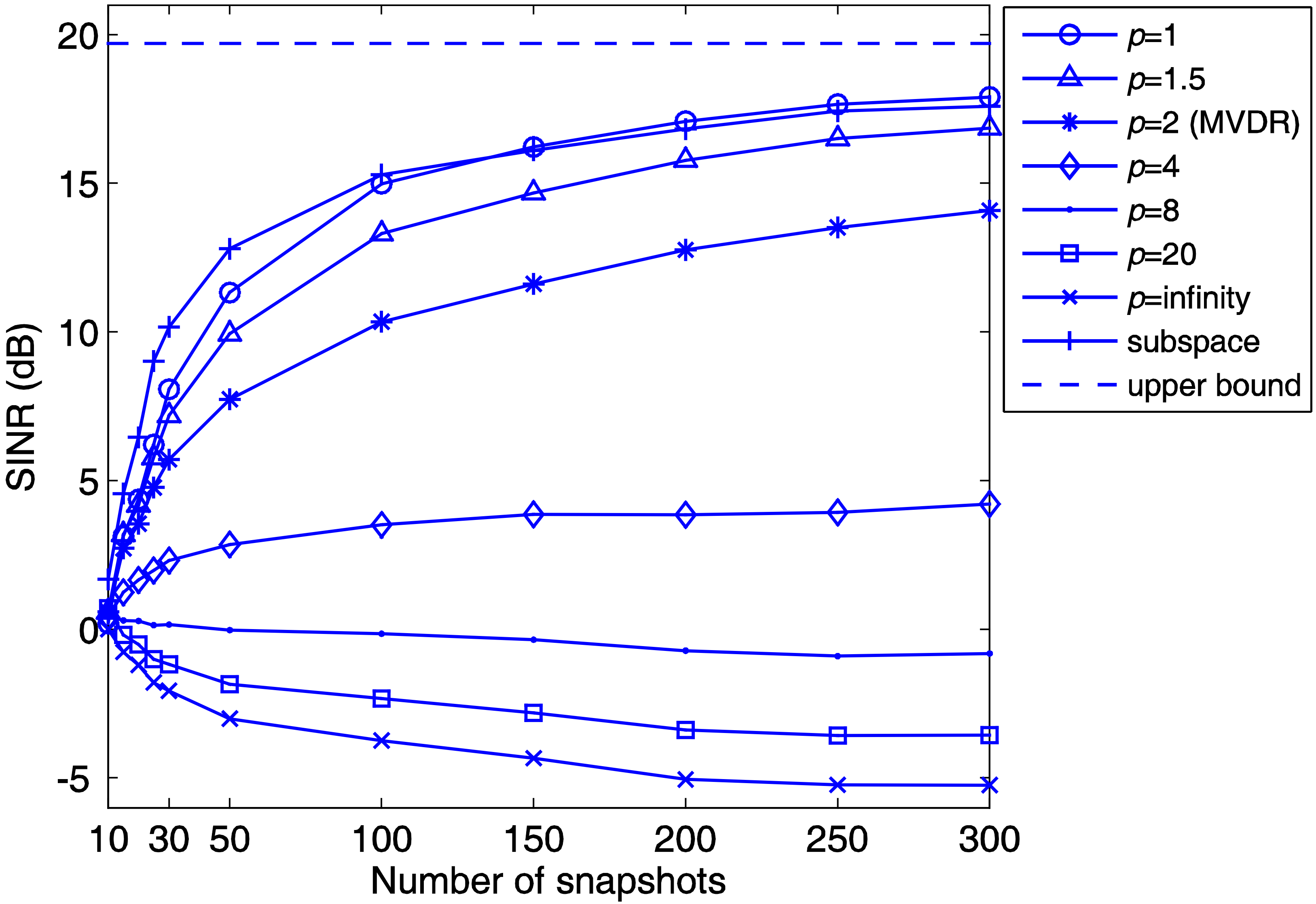
SINR versus SNR for QPSK sources in Gaussian noise



SINR versus N for QPSK sources in Gaussian noise



SINR versus SNR for super-Gaussian sources and noise



SINR versus N for super-Gaussian sources and noise

2. Linearly Constrained Minimum Dispersion (LCMD) [1]

The beamformer formulation is:

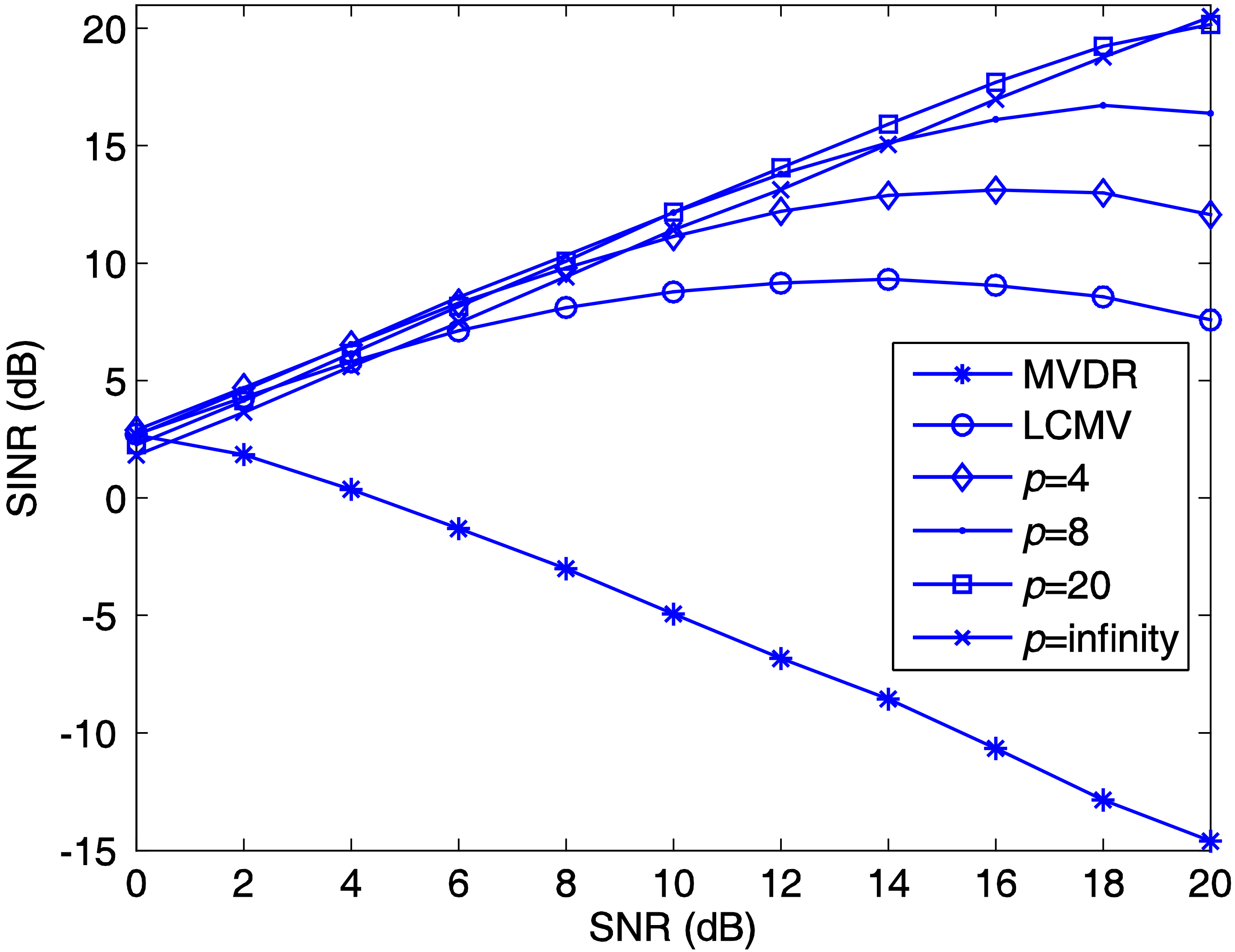
$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad \text{s.t. } \boldsymbol{\Phi}^H \boldsymbol{w} = \boldsymbol{g}, \quad p > 1$$

Applying iteratively reweighted idea, the solution at the kth iteration is:

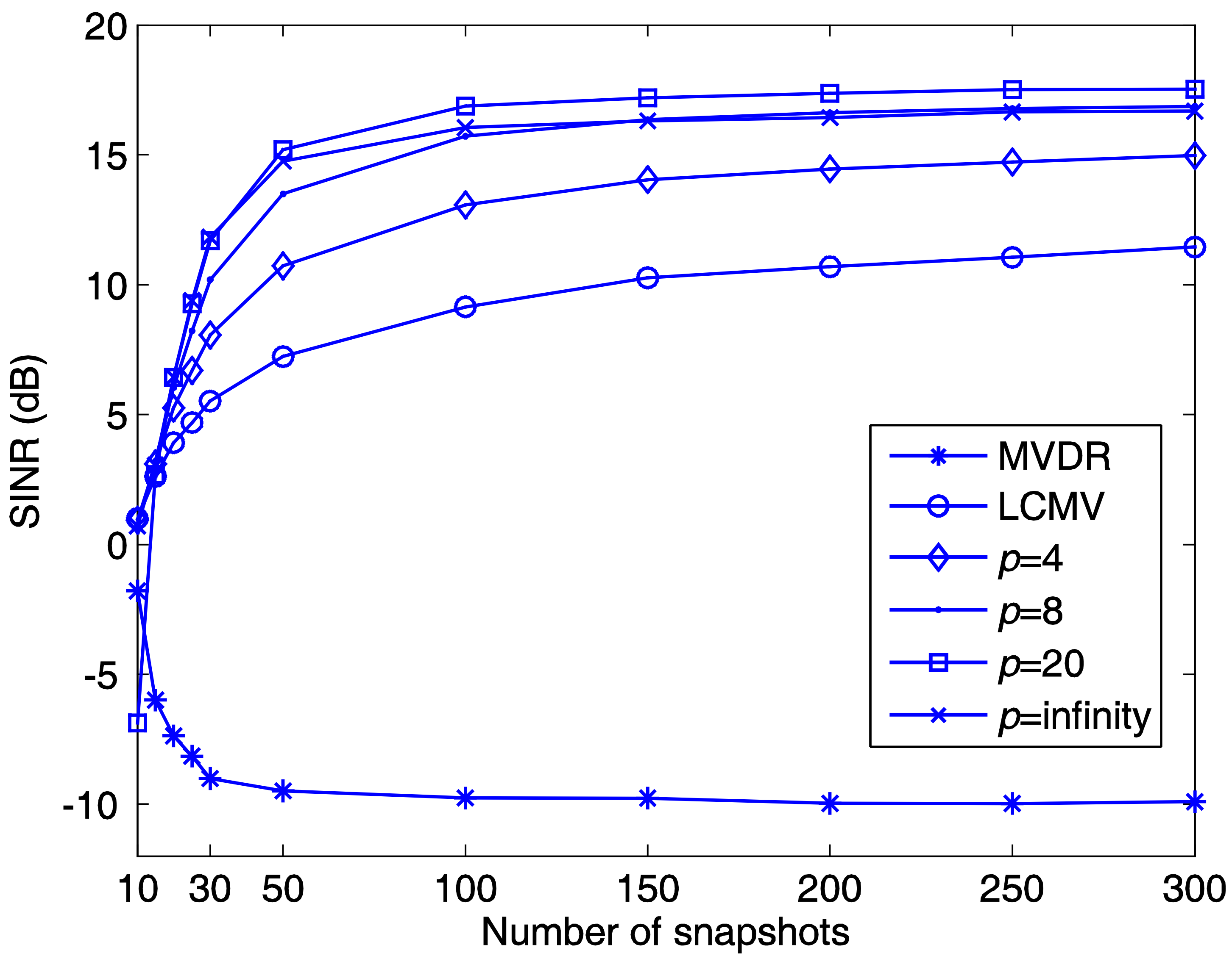
$$\boldsymbol{w}^{k+1} = \left(\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w}^k)\boldsymbol{X}^H\right)^{-1} \boldsymbol{\Phi} \left(\boldsymbol{\Phi}^H \left(\boldsymbol{X}\boldsymbol{D}(\boldsymbol{w}^k)\boldsymbol{X}^H\right)^{-1} \boldsymbol{\Phi}\right)^{-1} \boldsymbol{g}$$

Again, the more advanced technique is to use complex-valued Newton method [1].

The ℓ_∞-norm LCMD can also be cast as an SOCP as in the ℓ_∞ -norm MDDR.



SINR versus SNR for QPSK sources with DOA mismatch



SINR versus N for QPSK sources with DOA mismatch

For super-Gaussian SOI, interferences and noise, the MDDR beamformer cannot attain optimality even at $p=1$.

This motivates us to address:

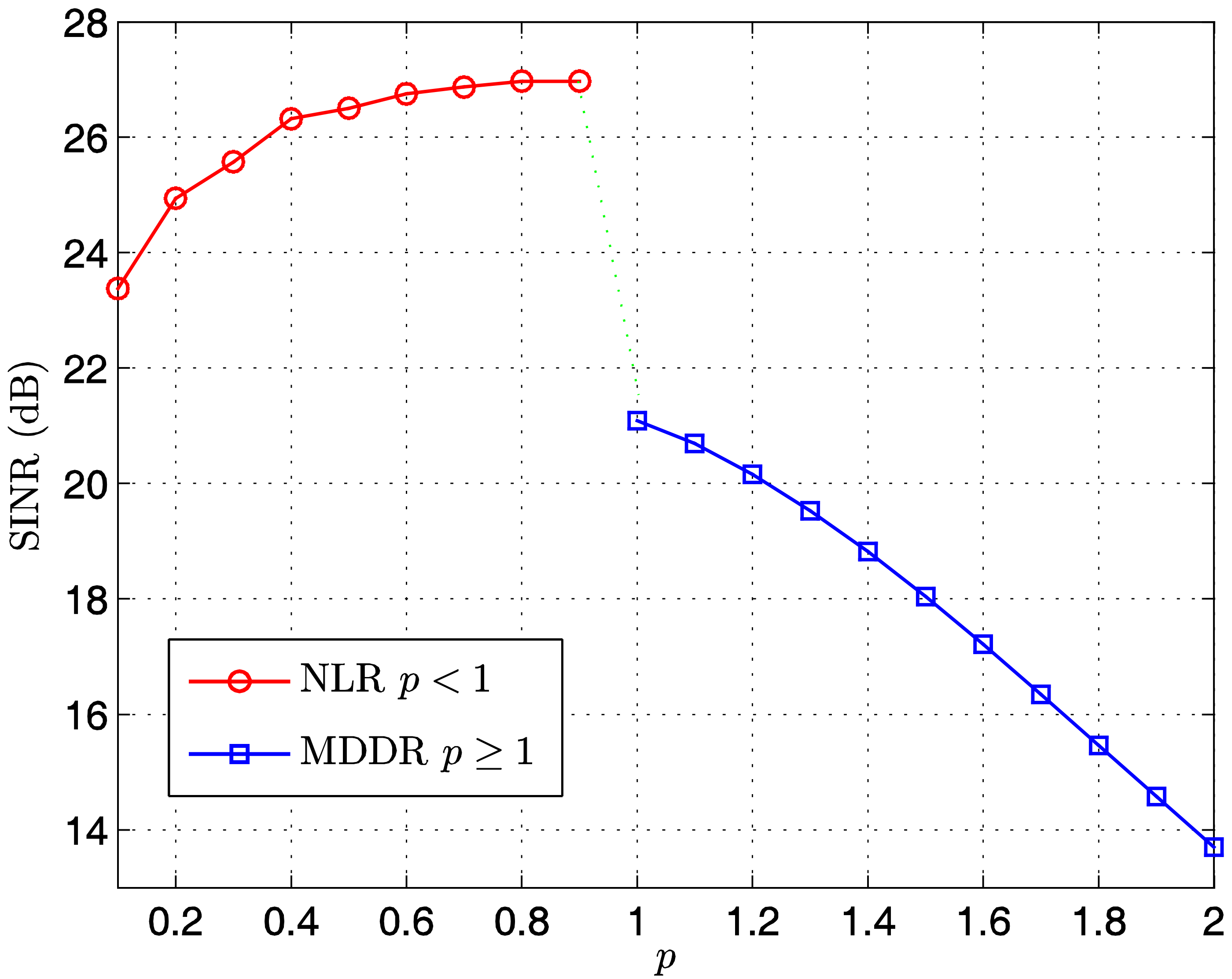
$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad \text{s.t. } \boldsymbol{w}^H \boldsymbol{a} = 1, \quad p < 1$$

and extend the LCMD formulation to:

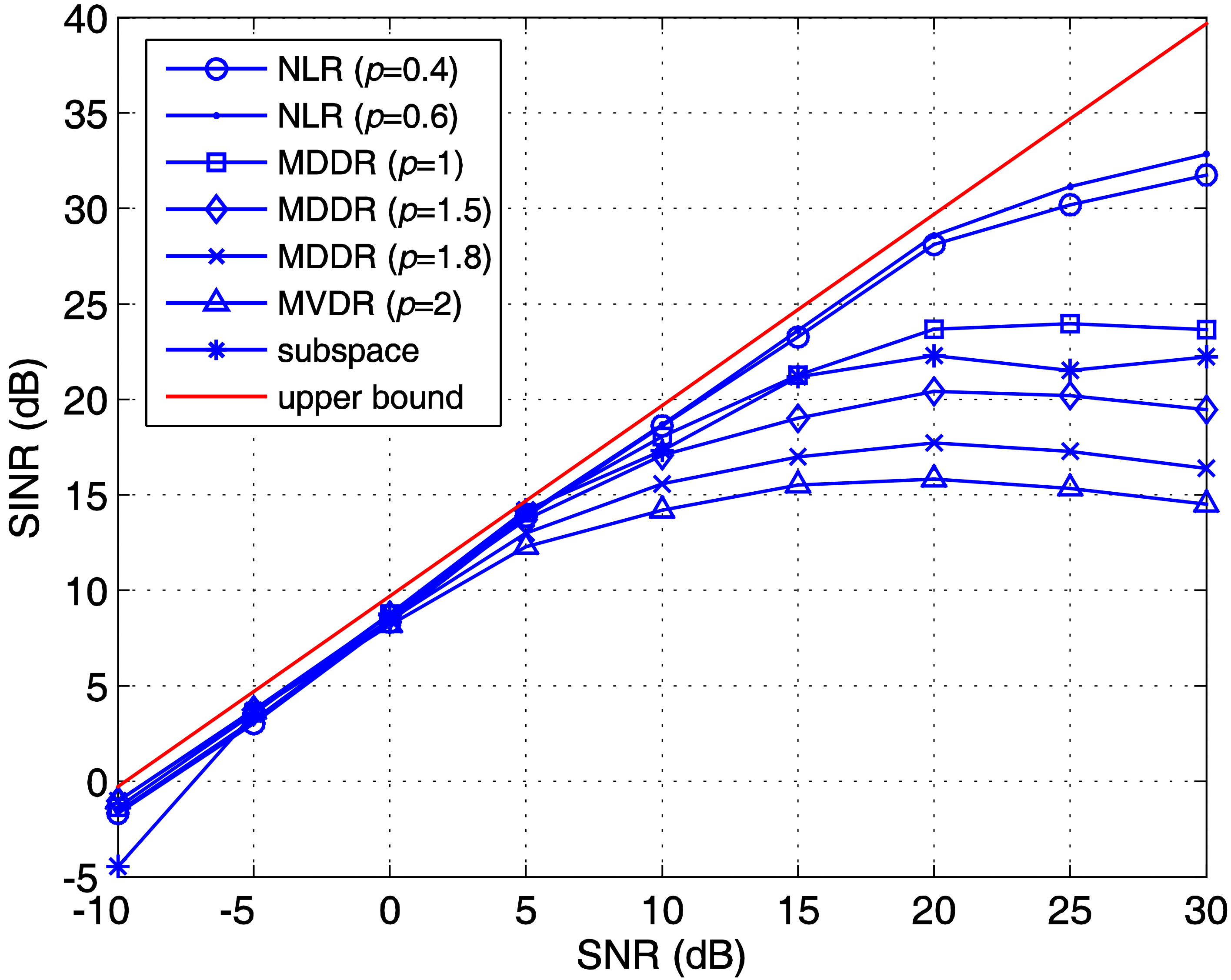
$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad \text{s.t. } \boldsymbol{\Phi}^H \boldsymbol{w} = \boldsymbol{g}, \quad p < 1$$

where the problems are now nonconvex and nonsmooth.

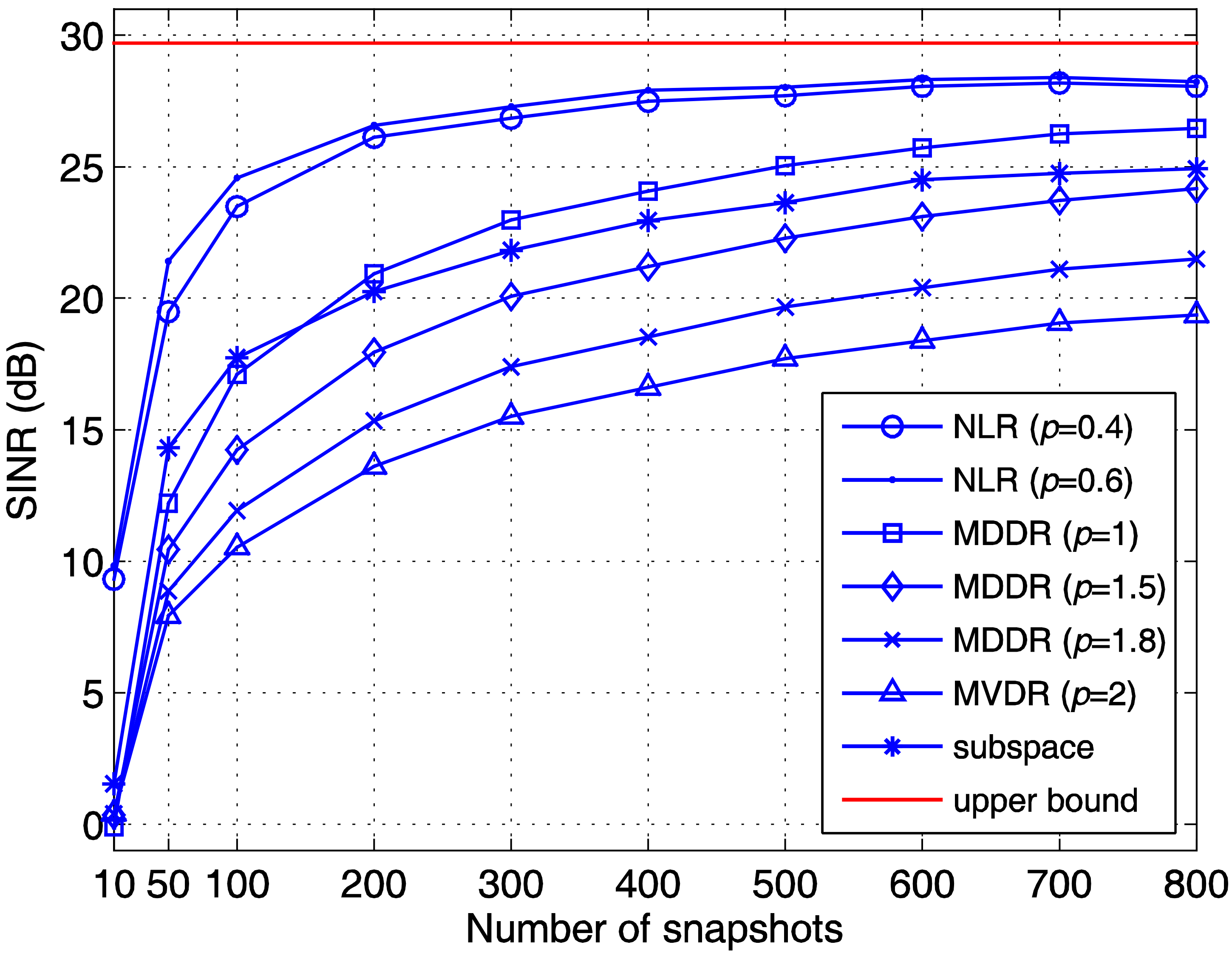
Applying the idea of coordinate descent, nonconvex linear regression (NLR) based beamforming algorithms which guarantee local convergence are devised [2].



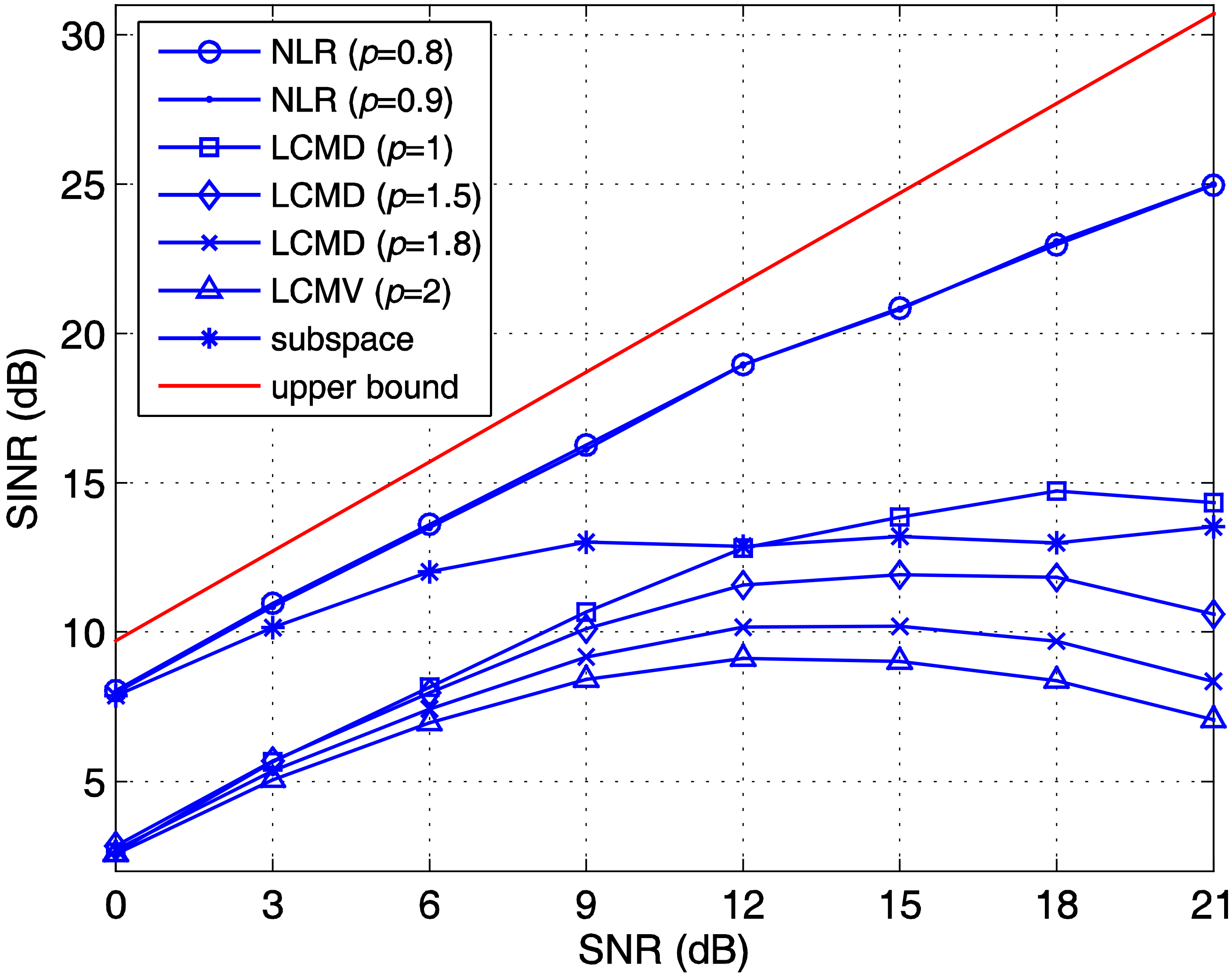
SINR versus p for super-Gaussian sources and noise



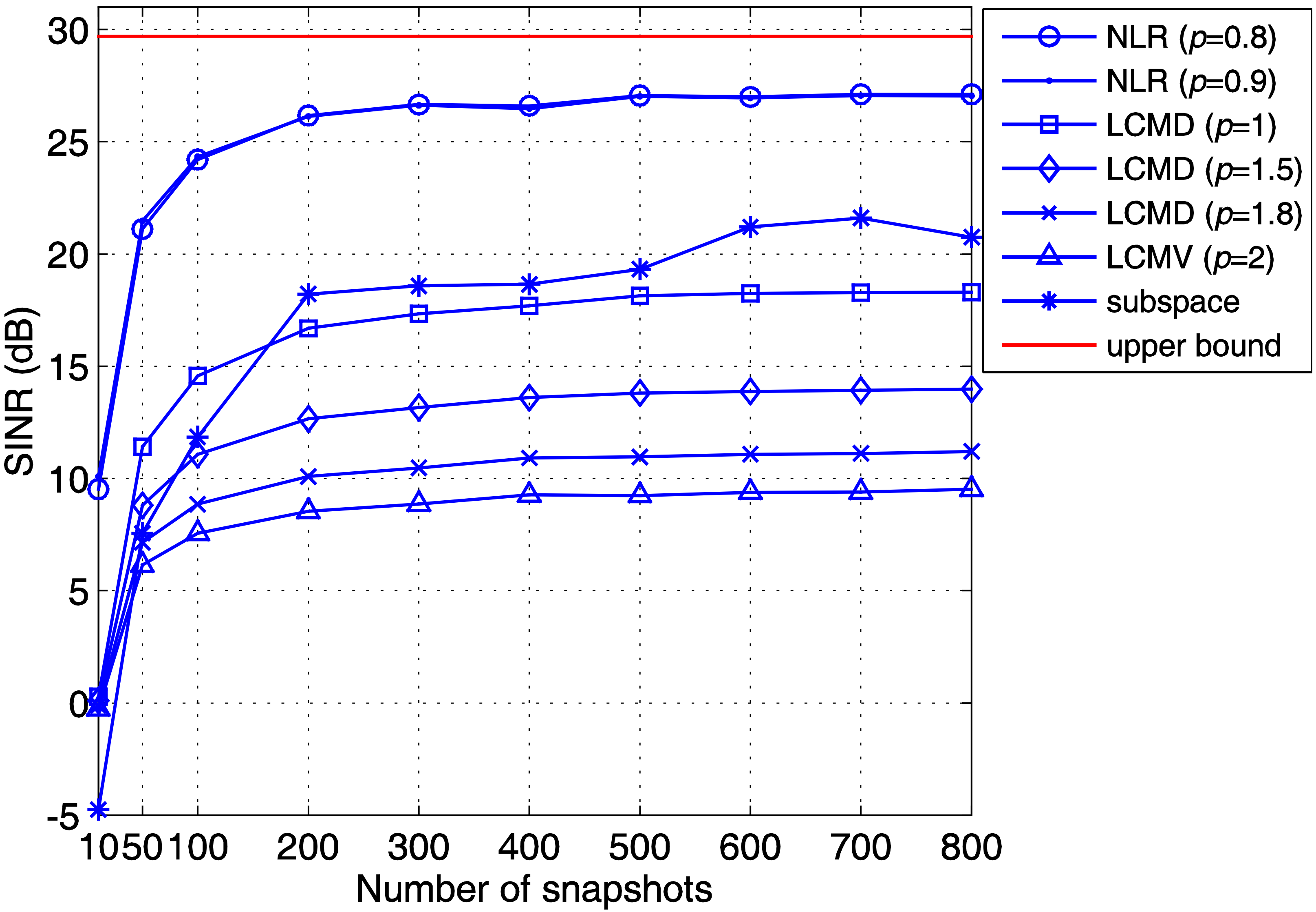
SINR versus SNR for super-Gaussian sources and noise



SINR versus N for super-Gaussian sources and noise



SINR versus SNR for super-Gaussian data with DOA mismatch



SINR versus N for super-Gaussian data with DOA mismatch

3. Minimum Dispersion based Worst-Case Performance Optimization with Quadratic Constraints [3]

Assume the uncertainty of steering error vector is bounded by a sphere $\mathcal{E} = \{\boldsymbol{e} \mid \|\boldsymbol{e}\| \le \varepsilon\}$, the beamformer formulation is:

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_p^p, \quad \text{s.t. } \mathrm{Re}(\boldsymbol{a}^H \boldsymbol{w}) \ge \varepsilon \|\boldsymbol{w}\| + 1$$

For $p \ge 1$, the objective is convex while the constraint constitutes a convex set, and thus convex optimization techniques such as interior point method can be employed.

Nevertheless, projected gradient methods (PGMs) [3] with closed-form projection are preferred because of their low computational requirement.

PGMs can be extended for the ellipsoidal uncertainty [3].

At $p = \infty$, we have:

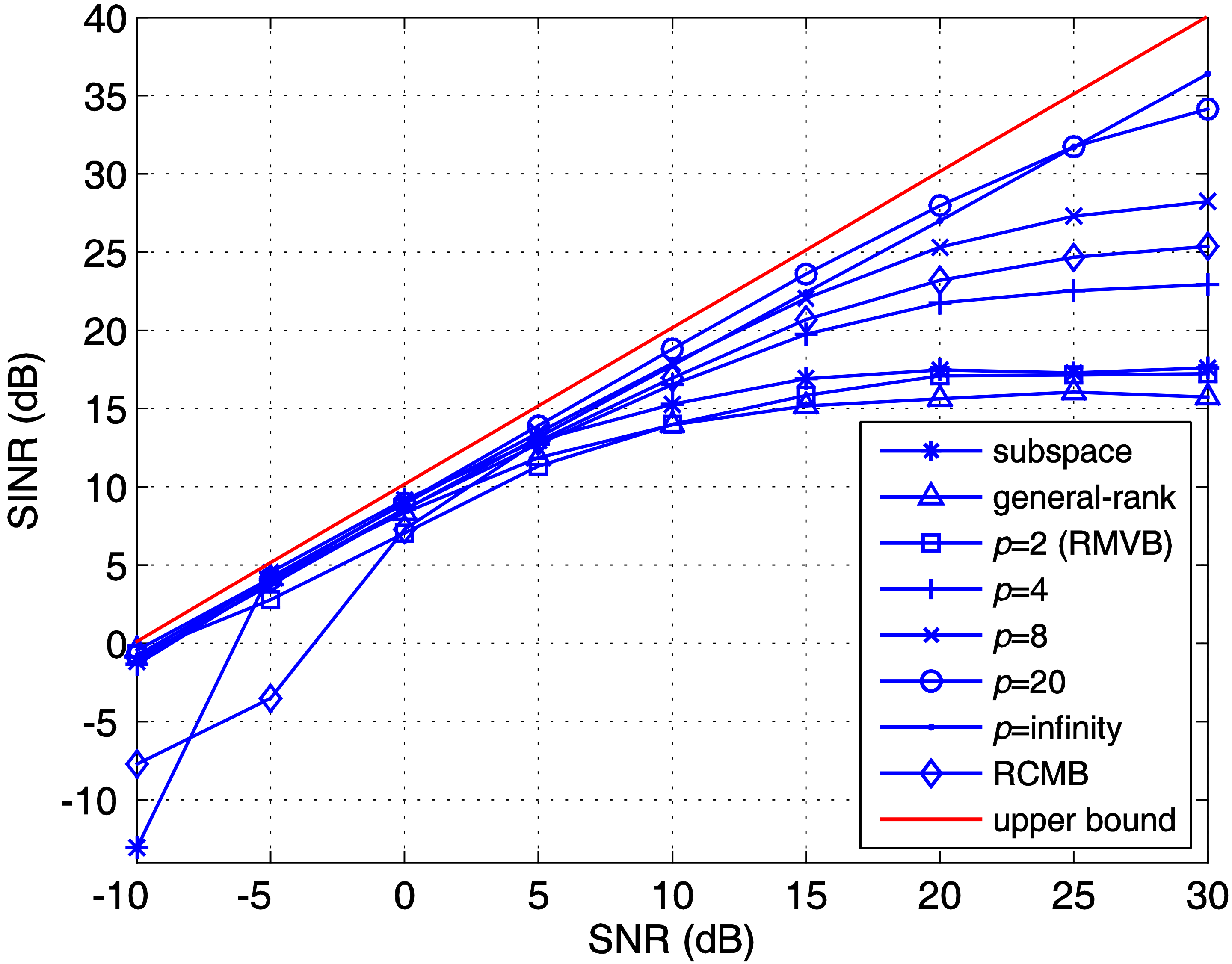
$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_\infty, \quad \text{s.t. } \mathrm{Re}(\boldsymbol{a}^H \boldsymbol{w}) \geq \varepsilon \|\boldsymbol{w}\| + 1$$

which can be converted as SOCP:

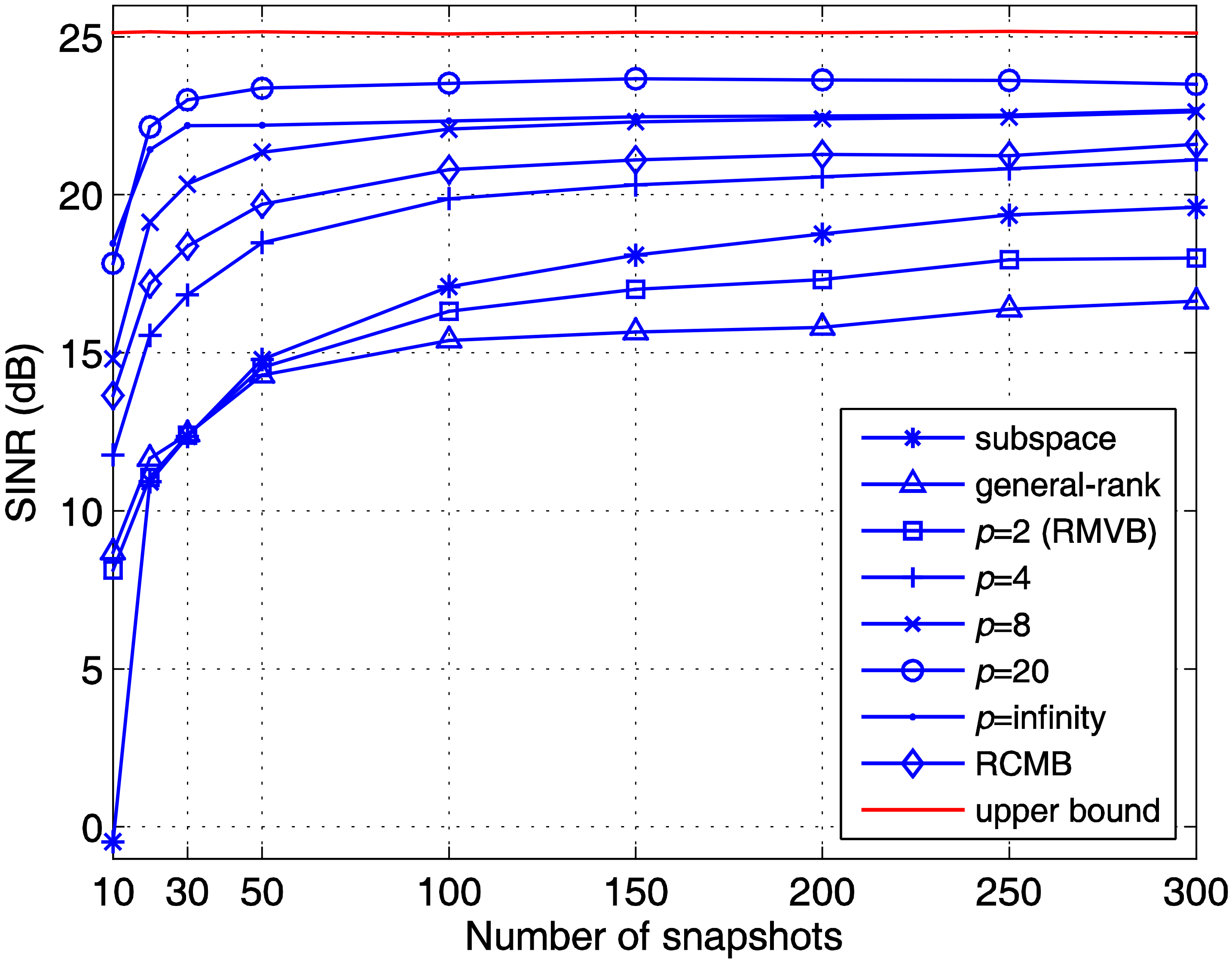
$$\begin{aligned} &\min_{\bar{\boldsymbol{w}}, \boldsymbol{y}_R, \boldsymbol{y}_I, t} \quad t \\ &\text{s.t. } \sqrt{y_R^2(n) + y_I^2(n)} \leq t, \quad n = 1, \cdots, N \\ &\qquad \begin{bmatrix} \boldsymbol{X}_R & \boldsymbol{X}_I \\ \boldsymbol{X}_I & -\boldsymbol{X}_R \end{bmatrix}^T \bar{\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{y}_R \\ \boldsymbol{y}_I \end{bmatrix} \\ &\qquad \bar{\boldsymbol{a}}^T \bar{\boldsymbol{w}} \geq \varepsilon \|\bar{\boldsymbol{w}}\| + 1 \end{aligned}$$

where

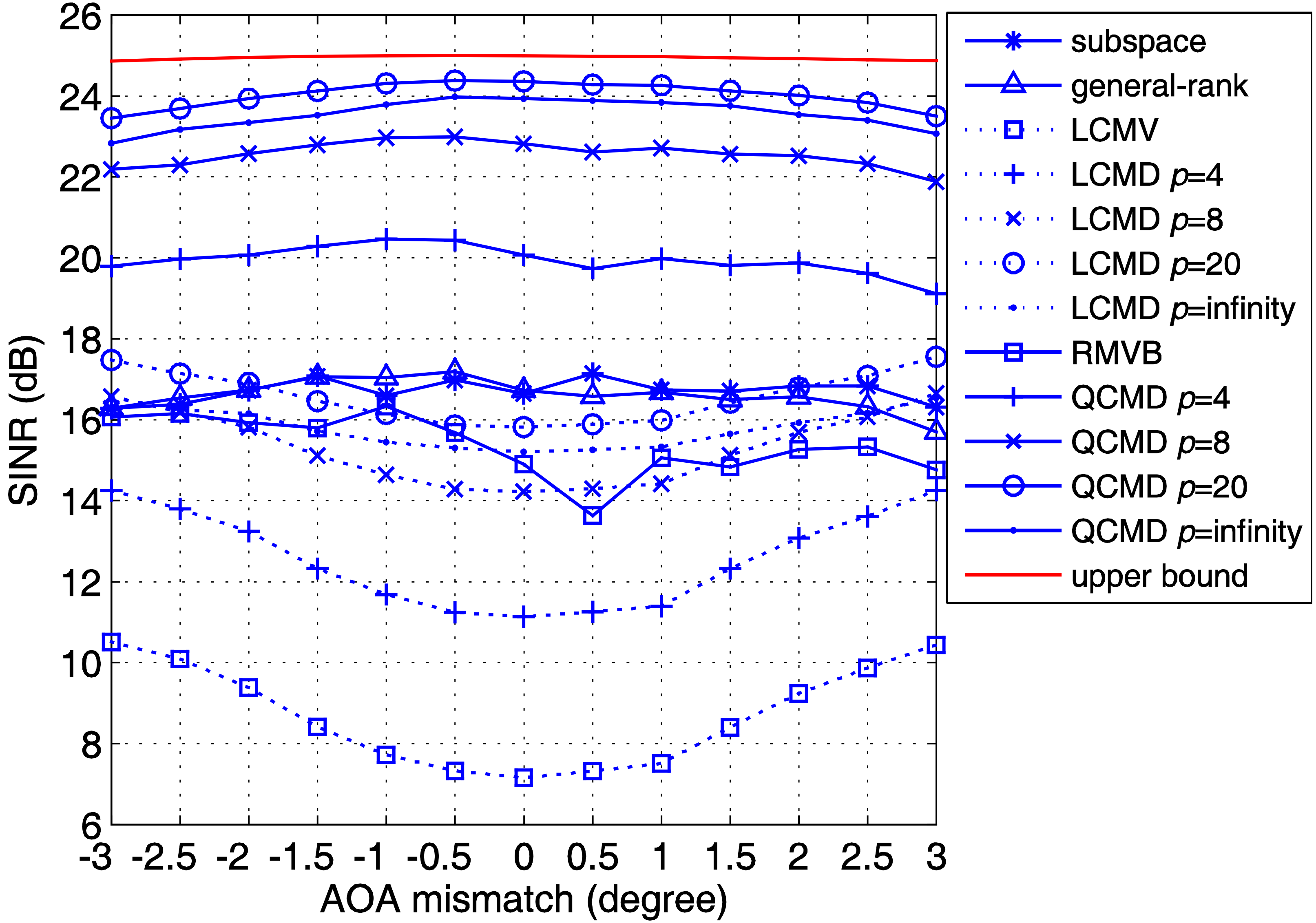
$$t \in \mathbb{R}^+, \quad \bar{\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{w}_R \\ \boldsymbol{w}_I \end{bmatrix} \quad \text{and} \quad \bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{a}_R \\ \boldsymbol{a}_I \end{bmatrix}$$



SINR versus SNR for QPSK sources with random mismatch



SINR versus N for QPSK sources with random mismatch



SINR versus DOA mismatch for QPSK sources

4. Minimum Dispersion based Worst-Case Performance Optimization with Linear Programming [4]

Here we focus on handling sub-Gaussian signals and ℓ_∞-norm is considered.

The beamformer formulation is:

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_\infty, \quad \text{s.t. } |(\boldsymbol{a}+\boldsymbol{e})^H \boldsymbol{w}|_\infty \geq 1, \text{ for all } \boldsymbol{e} \in \mathcal{E}$$

Two additional novel features are:

- $|(\boldsymbol{a}+\boldsymbol{e})^H \boldsymbol{w}|_\infty \geq 1$, corresponding to minmax approach, is employed instead of $|(\boldsymbol{a}+\boldsymbol{e})^H \boldsymbol{w}| \geq 1$.
- $\mathcal{E} = \{\boldsymbol{e} | \, \|\boldsymbol{e}\|_1 \leq \varepsilon\}$, implying that $\mathcal{E}$ is a rhombus.

However, there are infinitely many nonconvex constraints.

The ℓ_p-norm of $\boldsymbol{z} = \boldsymbol{z}_R + \mathrm{j}\boldsymbol{z}_I = [z_1 \ \cdots \ z_M]^T \in \mathbb{C}^M$ is defined as:

$$\|\boldsymbol{z}\|_p = \left(\sum_{m=1}^{M} |z_m|_p^p \right)^{\frac{1}{p}} = \left(\sum_{m=1}^{M} |\mathrm{Re}(z_m)|^p + |\mathrm{Im}(z_m)|^p \right)^{\frac{1}{p}}$$

In particular:

$$\|\boldsymbol{z}\|_1 = \sum_{m=1}^{M} [|\mathrm{Re}(z_m)| + |\mathrm{Im}(z_m)|] = \left\| \left[\boldsymbol{z}_R^T \ \boldsymbol{z}_I^T \right]^T \right\|_1$$

and

$$\|\boldsymbol{z}\|_\infty = \max_{1 \le m \le M} |z_m|_\infty = \max_{1 \le m \le M} (\max(|\mathrm{Re}(z_m)|, |\mathrm{Im}(z_m)|)) = \left\| \left[\boldsymbol{z}_R^T \ \boldsymbol{z}_I^T \right]^T \right\|_\infty$$

By the triangle inequality, we obtain

$$\left|\boldsymbol{a}^H\boldsymbol{w}+\boldsymbol{e}^H\boldsymbol{w}\right|_\infty \geq \left|\boldsymbol{a}^H\boldsymbol{w}\right|_\infty - \left|\boldsymbol{e}^H\boldsymbol{w}\right|_\infty$$

Denoting $\boldsymbol{a}=\boldsymbol{a}_R+\mathrm{j}\boldsymbol{a}_I$, $\boldsymbol{e}=\boldsymbol{e}_R+\mathrm{j}\boldsymbol{e}_I$ and $\boldsymbol{w}=\boldsymbol{w}_R+\mathrm{j}\boldsymbol{w}_I$, $\boldsymbol{e}^H\boldsymbol{w}$ is

$$\boldsymbol{e}^H\boldsymbol{w}=\boldsymbol{e}_R^T\boldsymbol{w}_R+\boldsymbol{e}_I^T\boldsymbol{w}_I+\mathrm{j}\left(-\boldsymbol{e}_I^T\boldsymbol{w}_R+\boldsymbol{e}_R^T\boldsymbol{w}_I\right)$$

$$\Rightarrow \left|\boldsymbol{e}^H\boldsymbol{w}\right|_\infty = \left\|\begin{bmatrix}\boldsymbol{e}_R^T\boldsymbol{w}_R+\boldsymbol{e}_I^T\boldsymbol{w}_I\\ -\boldsymbol{e}_I^T\boldsymbol{w}_R+\boldsymbol{e}_R^T\boldsymbol{w}_I\end{bmatrix}\right\|_\infty = \|\boldsymbol{E}^T\bar{\boldsymbol{w}}\|_\infty$$

where

$$\boldsymbol{E}=\begin{bmatrix}\boldsymbol{e}_R & -\boldsymbol{e}_I\\ \boldsymbol{e}_I & \boldsymbol{e}_R\end{bmatrix}\in\mathbb{R}^{2M\times 2} \quad \text{and} \quad \bar{\boldsymbol{w}}=\begin{bmatrix}\boldsymbol{w}_R\\ \boldsymbol{w}_I\end{bmatrix}\in\mathbb{R}^{2M}$$

Applying the matrix norm inequality yields

$$\left|\boldsymbol{e}^H\boldsymbol{w}\right|_\infty = \|\boldsymbol{E}^T\bar{\boldsymbol{w}}\|_\infty \leq \|\boldsymbol{E}\|_1\|\bar{\boldsymbol{w}}\|_\infty$$

where $\|\boldsymbol{E}\|_1$ is the maximum column sum matrix norm of $\boldsymbol{E}$.

$\|\boldsymbol{E}\|_1$ is bounded as

$$\|\boldsymbol{E}\|_1 = \max\left(\left\|\begin{bmatrix}\boldsymbol{e}_R\\ \boldsymbol{e}_I\end{bmatrix}\right\|_1, \left\|\begin{bmatrix}-\boldsymbol{e}_I\\ \boldsymbol{e}_R\end{bmatrix}\right\|_1\right) = \left\|\begin{bmatrix}\boldsymbol{e}_R\\ \boldsymbol{e}_I\end{bmatrix}\right\|_1 = \|\boldsymbol{e}\|_1 \leq \varepsilon$$

We then obtain:

$$\left|\boldsymbol{e}^H\boldsymbol{w}\right|_\infty \leq \varepsilon\|\bar{\boldsymbol{w}}\|_\infty$$

On the other hand, $\left|\boldsymbol{a}^H\boldsymbol{w}\right|_\infty$ is bounded from below by

$$\begin{aligned}\left|\boldsymbol{a}^H\boldsymbol{w}\right|_\infty &= \max\left(\left|\mathrm{Re}\left(\boldsymbol{a}^H\boldsymbol{w}\right)\right|, \left|\mathrm{Im}\left(\boldsymbol{a}^H\boldsymbol{w}\right)\right|\right)\\ &\geq \mathrm{Re}\left(\boldsymbol{a}^H\boldsymbol{w}\right) = \boldsymbol{a}_R^T\boldsymbol{w}_R + \boldsymbol{a}_I^T\boldsymbol{w}_I = \bar{\boldsymbol{a}}^T\bar{\boldsymbol{w}}\end{aligned}$$

Combining the results yields:

$$\left|\boldsymbol{a}^H\boldsymbol{w} + \boldsymbol{e}^H\boldsymbol{w}\right|_\infty \geq \bar{\boldsymbol{a}}^T\bar{\boldsymbol{w}} - \varepsilon\|\bar{\boldsymbol{w}}\|_\infty$$

Finally,

$$\min_{\boldsymbol{w}} \|\boldsymbol{X}^H \boldsymbol{w}\|_\infty, \quad \text{s.t. } |(\boldsymbol{a}+\boldsymbol{e})^H \boldsymbol{w}|_\infty \geq 1, \text{ for all } \boldsymbol{e} \in \mathcal{E} = \{\boldsymbol{e}| \, \|\boldsymbol{e}\|_1 \leq \varepsilon\}$$

is converted to

$$\min_{\bar{\boldsymbol{w}}} \ \|\bar{\boldsymbol{X}}^T \bar{\boldsymbol{w}}\|_\infty \quad \text{s.t. } \bar{\boldsymbol{a}}^T \bar{\boldsymbol{w}} - \varepsilon \|\bar{\boldsymbol{w}}\|_\infty \geq 1$$

where

$$\bar{\boldsymbol{X}} = \begin{bmatrix} \boldsymbol{X}_R & -\boldsymbol{X}_I \\ \boldsymbol{X}_I & \boldsymbol{X}_R \end{bmatrix}$$

which is convex optimization problem because the objective is a convex function and constraint is a convex set.

The convex constraint is stricter than that of original problem, i.e., the feasible region of latter is larger.

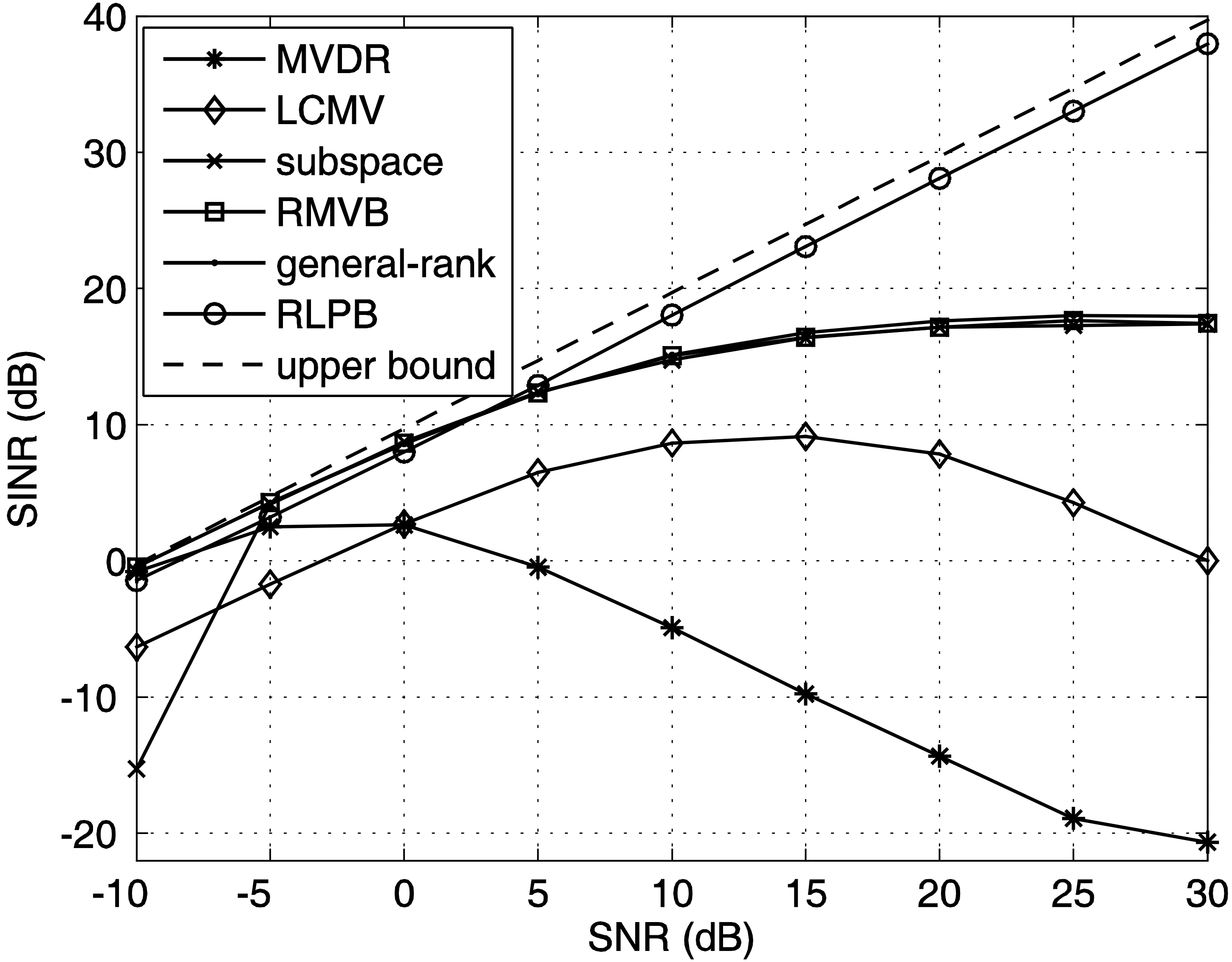
By introducing $u, r \in \mathbb{R}$, we obtain a linear program:

$$\min_{\bar{\boldsymbol{w}},u,r} \ u$$
$$\text{s.t.} \ -u\mathbf{1}_{2N} \leq \bar{\boldsymbol{X}}^T \bar{\boldsymbol{w}} \leq u\mathbf{1}_{2N}$$
$$\bar{\boldsymbol{a}}^T \bar{\boldsymbol{w}} \geq \varepsilon r + 1$$
$$-r\mathbf{1}_{2M} \leq \bar{\boldsymbol{w}} \leq r\mathbf{1}_{2M}$$

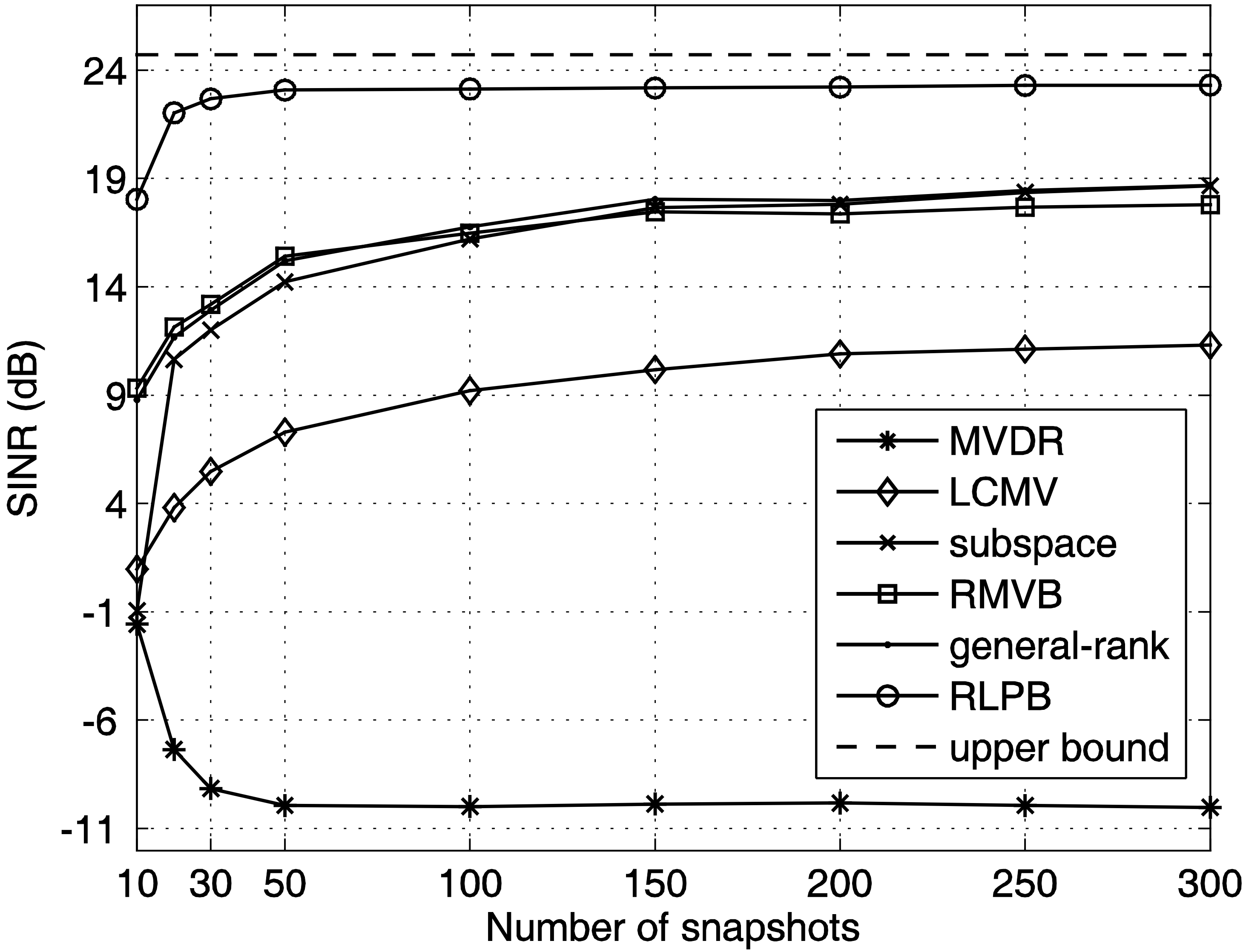
or its standard form:

$$\min_{\bar{\boldsymbol{w}},u,r} \ [\mathbf{0}_{2M}^T \ 1 \ 0] \begin{bmatrix} \bar{\boldsymbol{w}} \\ u \\ r \end{bmatrix}$$

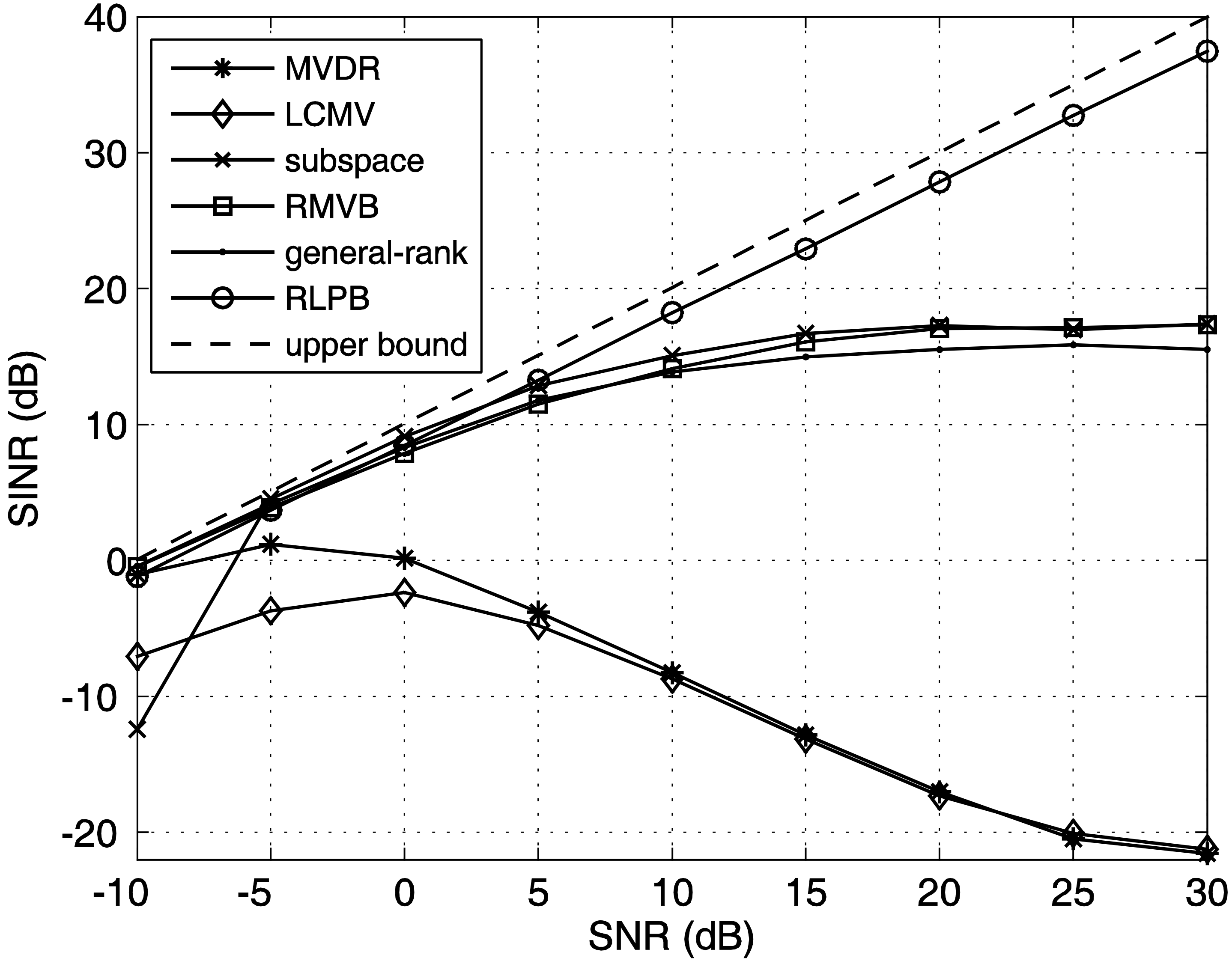
$$\text{s.t.} \ \begin{bmatrix} \bar{\boldsymbol{X}}^T & -\mathbf{1}_{2N} & \mathbf{0}_{2N} \\ -\bar{\boldsymbol{X}}^T & -\mathbf{1}_{2N} & \mathbf{0}_{2N} \\ \boldsymbol{I}_{2M\times 2M} & \mathbf{0}_{2M} & -\mathbf{1}_{2M} \\ -\boldsymbol{I}_{2M\times 2M} & \mathbf{0}_{2M} & -\mathbf{1}_{2M} \\ -\bar{\boldsymbol{a}}^T & 0 & \varepsilon \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{w}} \\ u \\ r \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}_{2N} \\ \mathbf{0}_{2N} \\ \mathbf{0}_{2M} \\ \mathbf{0}_{2M} \\ -1 \end{bmatrix}$$



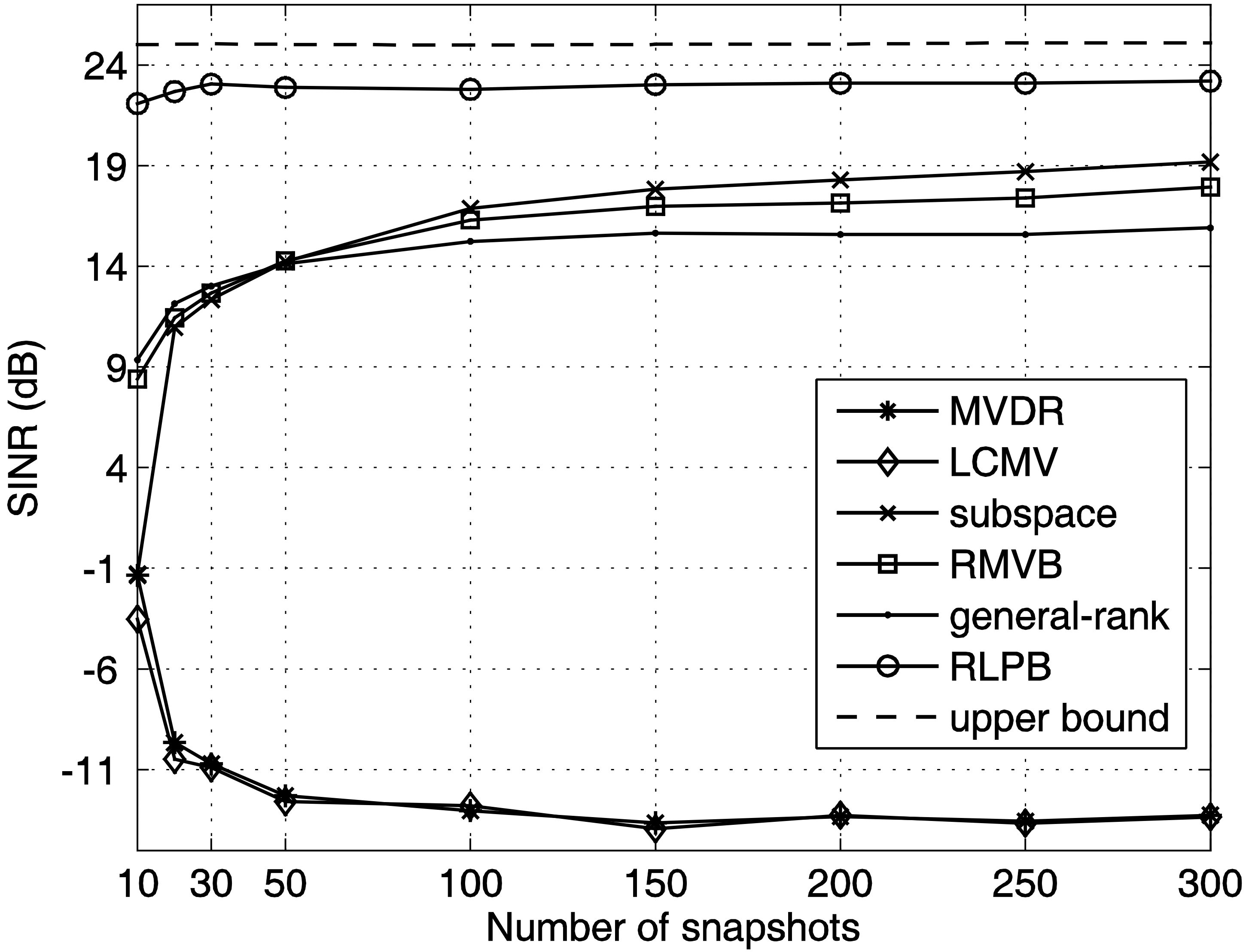
SINR versus SNR for QPSK sources with DOA mismatch



SINR versus N for QPSK sources with DOA mismatch



SINR versus SNR for QPSK sources with random mismatch



SINR versus N for QPSK sources with random mismatch

Algorithm	Mismatch handled	Complexity/iteration
MDDR ($\infty > p \geq 1$)	-	$\mathcal{O}(MN)$
MDDR ($p = \infty$)	-	$\mathcal{O}(N^3)$
MDDR ($p < 1$)	-	$\mathcal{O}(MN^2)$ for local min.
LCMD ($\infty > p \geq 1$)	DOA	$\mathcal{O}(MN)$
LCMD ($p = \infty$)	DOA	$\mathcal{O}(N^3)$
LCMD ($p < 1$)	DOA	$\mathcal{O}(MN^2)$ for local min.
QCMD ($\infty > p \geq 1$)	Arbitrary	$\mathcal{O}(MN)$
QCMD ($p = \infty$)	Arbitrary	$\mathcal{O}(N^3)$
RLPB ($p = \infty$)	Arbitrary	$\mathcal{O}(N^3)$

Summary of Robustness and Complexities

Summary

- Minimum dispersion criterion is devised which is a generalization of the minimum variance criterion from ℓ_2-norm to ℓ_p-norm where $p \in (0, \infty]$.

- For sub-Gaussian sources, $p > 2$ or even $p = \infty$ is preferred. For super-Gaussian sources, $p < 2$ is preferred.

- For linear constraints, we have devised the MDDR and LCMD which generalize the MVDR and LCMV, respectively.

- Based on worst-case performance optimization approach, we have devised two approaches: the first results with quadratic constraints while another is cast as a linear program.

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[3] X. Jiang, W.-J. Zeng, A. Yasotharan, H.C. So and T. Kirubarajan, "Quadratically constrained minimum dispersion beamforming via gradient projection," *IEEE Transactions on Signal Processing*, vol.63, no.1, pp.192-205, Jan. 2015

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