# Generation of Correlated PSK Waveforms Using Complex Gaussian Random Variables

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## Objectives

- Derive the closed-form relationship between the cross-correlation of complex Gaussian random variables (RVs) and finite alphabets.
- Generate correlated PSK waveforms using complex Gaussian RVs
- Study the performance of correlated PSK signals in matching the desired transmit beampattern for the MIMO radar application

## Introduction

In MIMO radar beamforming, carefully designed correlated signals are transmitted from an array of collocated antennas to create constructive interference in the region of interest and destructive interference elsewhere. For practical reasons, the synthesized waveforms should have finite alphabets and low peak-to-average-power-ratio (PAPR).
Several works focused on designing infinite alphabet signals to match the desired beampettern. Besides, the authors in [1] designed finite alphabet waveforms drawn from conventional modulation schemes. The proposed method maps easily generated real Gaussian random variables (RVs) onto phase-shift-keying (PSK) or pulse-amplitude-modulation (PAM) symbols.
The main contribution of this work is to ameliorate the performance of higher PSK symbols by employing complex Gaussian RVs instead. This work relies on a new mapping function, better suited for the design of PSK symbols. The derived relationship between the correlation of complex Gaussian RVs and PSK symbols approaches the identity function and improves the beampattern matching performance.

## Problem Formulation

Consider a uniform linear array of $N$ collocated antenna elements with half wavelength inter-element spacing. The received power at an angle $\theta$ is given by $P(\theta) = \mathbf{e}^H(\theta)\mathbf{R}\mathbf{e}(\theta)$, where $\mathbf{R}$ is the covariance matrix of the transmitted waveforms and $\mathbf{e}(\theta)$ is the antenna's steering vector. Consequently, matching a desired transmit beampattern consists in designing signals with proper correlation properties.
Once $\mathbf{R}$ is synthesized, a trivial waveform solution can be generated using Gaussian RVs as follows $\mathbf{V} = \mathbf{R}^{1/2}\mathcal{V}$, where $\mathcal{V} \in \mathbb{C}^{N\times L}$ is a matrix of zero mean and unit variance complex Gaussian RVs. Although they are easily generated, Gaussian signals do not satisfy practical constraints. Indeed, they are not drawn from a finite alphabet and their PAPR, defined as $\text{PAPR}(\mathbf{v}_n) = \frac{\max_l |\mathbf{v}_n(l)|^2}{\frac{1}{L}\sum_{l=1}^{L}|\mathbf{v}_n(l)|^2}$, is quite high.

For these reasons, the easily generated Gaussian signals are mapped into constant modulus PSK symbols. This operation surely affects the correlation properties of the signal. Next, the relationship between the correlation of Gaussian RVs and PSK waveforms will be developed.

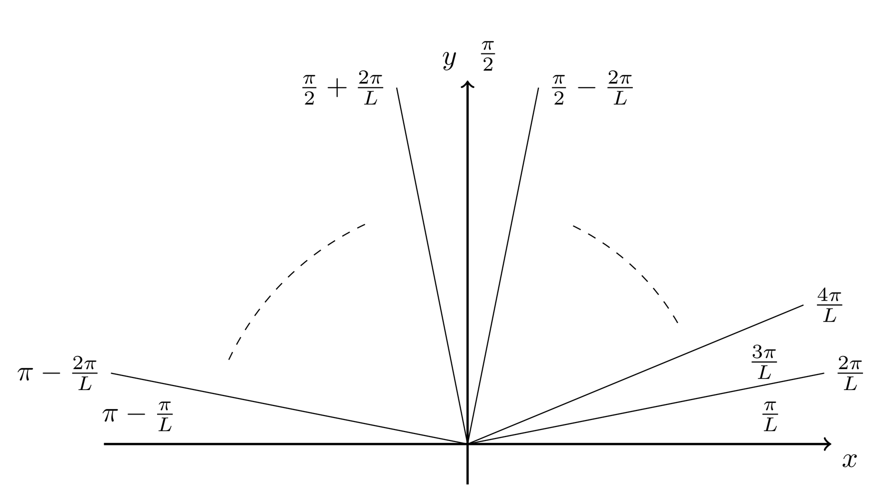


Figure 1: Illustration of the mapping function $f$.

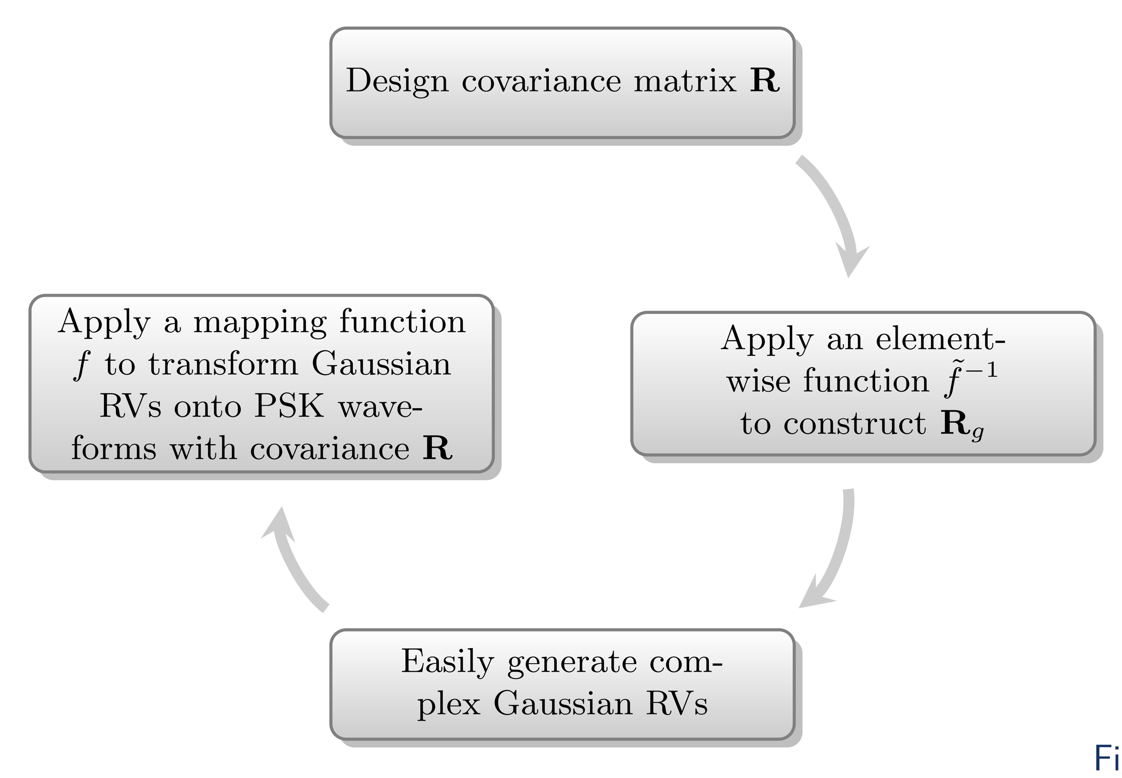


Figure 2: Illustration of the process to generate correlated PSK symbols using normal random variables.

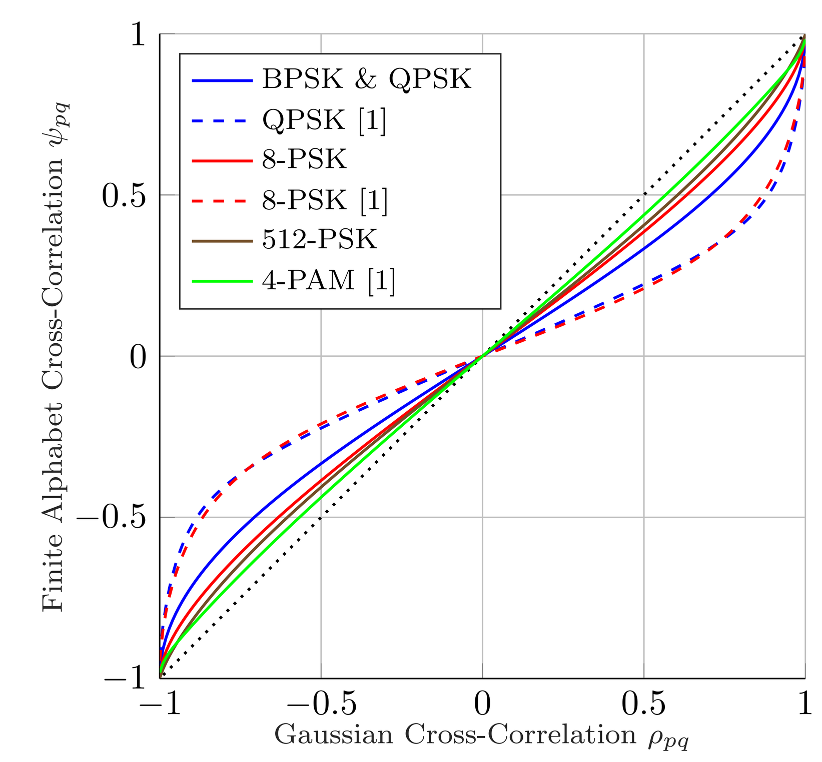


Figure 3: Relationship between the cross-correlation of Gaussian RVs and various modulation schemes. The dotted black graph corresponds to the identity function included for reference.

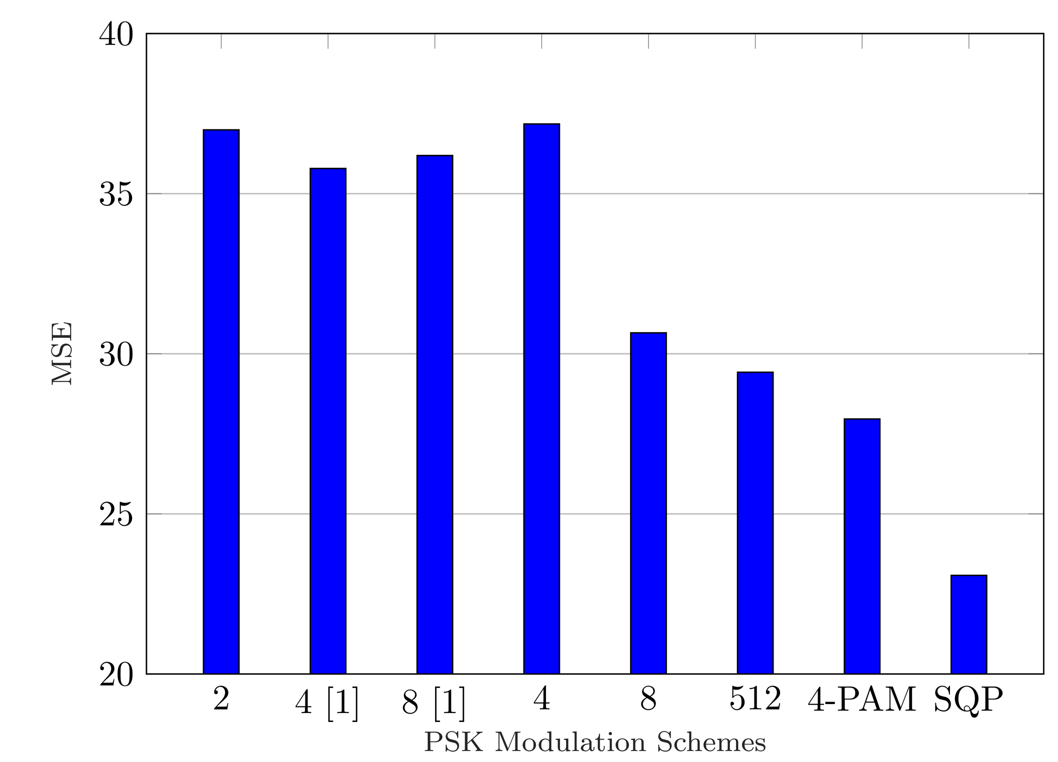


Figure 4: Beampattern matching performance of various PSK modulation schemes. SQP bar corresponds to the MSE achieved by the optimal covariance matrix.

## PSK Waveform Generation

Let $\tilde{f}$ be the relationship between the cross-correlation of complex Gaussian RVs $\rho_{pq}$ and finite alphabets $\psi_{pq}$. To express it as a polynomial function of $\rho_{pq}$, we extended the work in [2] to cover the case of complex Gaussian RVs. Using the physicists' Hermite polynomials $H_n(\cdot)$, the general expression can be reformulated as

$$\begin{aligned}\psi_{pq} = \tilde{f}(\rho_{pq}) &= \int_{\mathbb{C}}\int_{\mathbb{C}} w_p w_q^* \, p(v_p, v_q, \rho_{pq}) dv_p dv_q, \\ &= \frac{1}{\pi^2}\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\rho_{pq}^{n+m}}{2^{2n+m}n!m!}\times\left|\iint_{\mathbb{R}^2} f(x,y)H_n(x)H_m(y)\mathrm{e}^{-x^2}\mathrm{e}^{-y^2}dxdy\right|^2.\end{aligned} \tag{1}$$

There are numerous methods to convert Gaussian RVs into PSK symbols. The most straightforward way to create $L$-PSK waveforms is to assign a unique symbol to each region using the following mapping function

$$f(x,y) = \exp\left(j\frac{\pi}{L}(2l+1)\right), \text{ if } \angle x+jy \in \left[\frac{2\pi l}{L}, \frac{2\pi(l+1)}{L}\right], \tag{2}$$

where $l = -\frac{L}{2}, -\frac{L}{2}+1, \cdots, \frac{L}{2}-1$ and $\angle\,.$ returns the phase of the complex number. Thus, by applying the above mapping function, the general Taylor expansion of the relationship $\tilde{f}$ between the cross-correlation of complex Gaussian RVs and the generated $L$-PSK symbols is expressed as

$$\begin{aligned}\psi_{pq} = \frac{8}{\pi}\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\rho_{pq}^{2n+2m+1}}{2^{2n+2m+1}(2n+1)!(2m)!}\left(\frac{(2m+2n)!}{(m+n)!}\right)^2\times \\ \left|\sum_{l=0}^{L/4-1}\cos\left(\frac{\pi}{L}(2l+1)\right)\left(I_{n,m}\left(\frac{2\pi(l+1)}{L}\right) - I_{n,m}\left(\frac{2\pi l}{L}\right)\right)\right|^2,\end{aligned} \tag{3}$$

where $I_{n,m}(\alpha) = \cos^{2m}(\alpha)\sin^{2n+1}(\alpha)$.

## Examples and Relation to Previous Work

- Mapping complex Gaussian RVs into BPSK or QPSK signals results in the same relationship defined as

$$\psi_{pq} = \frac{2}{\pi}\,\text{asin}\left(\rho_{pq}\right). \tag{4}$$

- Mapping complex Gaussian RVs into 8-PSK signals results in the below relationship

$$\begin{aligned}\psi_{pq} &= \frac{8}{\pi}\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\rho_{pq}^{2n+2m+1}}{2^{2n+2m+1}(2n+1)!(2m)!}\left(\frac{(2m+2n)!}{(m+n)!}\right)^2\times\sin^2\left(\frac{\pi}{8}\right)\left(\delta_{0,m}+\frac{1}{2^{m+n}}\right)^2 \\ &= \frac{4}{\pi}\sin^2\left(\frac{\pi}{8}\right)\left(2\,\text{asin}\left(\rho_{pq}\right) + 2\sqrt{2}\,\text{asin}\left(\frac{\rho_{pq}}{\sqrt{2}}\right)\right)\end{aligned} \tag{5}$$

## Simulation Results

The complex Gaussian RVs are mapped into different $L$-PSK modulations. Their respective correlation functions $\tilde{f}$ are illustrated in Fig. 3. For comparison purposes, a selection of previous PAM and PSK results are also included. Contrarily to the previous work, it can be noticed that, as the number of alphabets increases, the relationship $\tilde{f}$ approaches the identity function.
Fig. 4 compares the MSE between a desired beampattern $\phi$ and the designed one for different modulation schemes. For this purpose, the region of interest is set to $[-30°, 30°]$ and the optimized cost function is defined as $\text{MSE} = \left\|\mathbf{e}^H(\theta)\mathbf{R}\mathbf{e}(\theta) - \phi(\theta)\right\|^2$. It can be noticed that 2-PSK and 4-PSK waveforms realize comparable MSE since they share the same correlation relationship. However, the performance of 8-PSK and 512-PSK signals produce a significant improvement.

## Conclusion

This work proposed a new method to generate correlated PSK symbols using complex Gaussian RVs. The general relationship between the correlation of Gaussian and finite waveforms is derived in closed form and presented as a Taylor series. We also discussed how the newly developed results are related to the previous work [1]. Due to limited space, we did not discuss in details the performance of the proposed waveforms for matching the desired beampattern. This topic, along with few other proofs, will be discussed in the full version of this manuscript.

## References

[1] S. Jardak, S. Ahmed, and M.-S. Alouini, "Generation of correlated finite alphabet waveforms using Gaussian random variables," *IEEE Transactions on Signal Processing*, vol. 62, pp. 4587–4596, Sept. 2014.

[2] J. Brown, J., "On the expansion of the bivariate Gaussian probability density using results of nonlinear theory," *IEEE Transactions on Information Theory*, vol. 14, pp. 158–159, Jan. 1968.

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