Fast Sparse Recovery via Non-Convex Optimization

Laming Chen and Yuantau Gu

Department of Electronic Engineering
Tsinghua University
Email: chen-lm06@mails.tsinghua.edu.cn, gyt@tsinghua.edu.cn
Homepage: http://gu.ee.tsinghua.edu.cn/

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Preliminary

Sparse Recovery Problem

- One tries to find a sparse solution to an underdetermined linear system.
- Using ℓ_1 norm to induce sparsity is a standard technique.
- Some certain non-convex functions tend to outperform ℓ_1 norm empirically in sparse recovery.

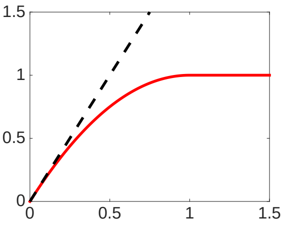
Motivation

- The convergence results of these non-convex algorithms are still very limited.
- We aim to devise a fast algorithm and to provide its convergence results.

Preliminary

Weak Convexity

- The non-convex $F(\cdot)$ becomes convex by adding a quadratic term.
- Let $\rho > 0$ be the smallest quantity such that $F(x) + \rho x^2$ is convex.
- These exists $\alpha > 0$ such that $F(x)/x \to \alpha$ as $x \to 0^+$.



$\frac{\rho}{\alpha}$: non-convexity of $F(\cdot)$

Preliminary

Problem Setup

- Consider the optimization problem

$$\arg\min_{\mathbf{x}} J(\mathbf{x}) + \frac{\tau}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$$

where $J(\mathbf{x}) = \sum_{i=1}^{N} F(x_i)$ is fully separable, and non-convex scalar function $F : \mathbb{R} \to \mathbb{R}^+$ satisfies:

(a) $F(0) = 0$, $F(\cdot)$ is even and not identically zero;
(b) $F(\cdot)$ is non-decreasing on $[0, +\infty)$;
(c) The function $x \mapsto F(x)/x$ is non-increasing on $(0, +\infty)$;
(d) $F(\cdot)$ is weakly convex on $[0, +\infty)$.

- Such $J(\mathbf{x})$ is common in sparse recovery literatures.

Concrete Examples of $F(\cdot)$

Requirements: $0 \leq p < 1$ and $\sigma > 0$

No.	$F(x)$	ρ	α
1.	$\frac{\lvert x\rvert}{(\lvert x\rvert+\sigma)^{1-p}}$	$(1-p)\sigma^{p-2}$	σ^{p-1}
2.	$1-\mathrm{e}^{-\sigma\lvert x\rvert}$	$\sigma^2/2$	σ
3.	$\ln(1+\sigma\lvert x\rvert)$	$\sigma^2/2$	σ
4.	$\mathrm{atan}(\sigma\lvert x\rvert)$	$3\sqrt{3}\sigma^2/16$	σ
5.	$(2\sigma\lvert x\rvert-\sigma^2x^2)\mathbf{1}_{\lvert x\rvert\leq\frac{1}{\sigma}}+\mathbf{1}_{\lvert x\rvert>\frac{1}{\sigma}}$	σ^2	2σ

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Algorithm

For the optimization problem

$$\arg\min_{\mathbf{x}} J(\mathbf{x}) + \frac{\tau}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$$

Algorithm 1 Proposed Algorithm

Require: $\mathbf{y}$, $\mathbf{A}$, $\tau > 0$, $\delta > 0$

Initialize: $l = 0$, $\mathbf{x}^0 = \mathbf{0}$;
while not converge **do**
  $\nabla^l = \mathbf{x}^l - \delta\mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{x}^l - \mathbf{y})$;
  $\mathbf{x}^{l+1} = \mathrm{prox}_J(\nabla^l, \delta/\tau)$ [1];
  $l = l + 1$;
end while

[1] $\mathrm{prox}_J(\mathbf{v}, \lambda) = \arg\min_{\mathbf{x}} J(\mathbf{x}) + \frac{1}{2\lambda}\|\mathbf{x} - \mathbf{v}\|_2^2$

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Convergence Results

Define the objective function

$$G(\mathbf{x}) = J(\mathbf{x}) + \frac{\tau}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2$$

Theorem 1. (Convergence to stationary point)

Assume sequence $\{\mathbf{x}^l\}$ is generated by Algorithm 1. If δ satisfies $0 < \delta < \min\{1/\|\mathbf{A}^T\mathbf{A}\|_2, \tau/(2\rho)\}$, then

(a) The sequence $\{G(\mathbf{x}^l)\}$ is non-increasing and convergent;

(b) Any limit point of $\{\mathbf{x}^l\}$ is a stationary point of the problem.

Convergence Results

Theorem 2. (Convergence to sparse signal)

(a) Assume $\|\mathbf{x}^*\|_0 \leq K$, $\mathbf{y} = \mathbf{A}\mathbf{x}^* + \mathbf{e}$, and $M_0 = \|\mathbf{x}^0 - \mathbf{x}^*\|_2$;

(b) Suppose $\gamma(J, \mathbf{A}, K) < 1$ and the non-convexity $\rho/\alpha \leq c_1/M_0$;

(c) Suppose $0 < \delta < \min\{1/(\|\mathbf{A}^T\mathbf{A}\|_2 + \|\mathbf{y}\|_2^2/\|\mathbf{x}^*\|_2^2), \tau/(2\rho)\}$, and the regularization parameter $\tau = c_2/\|\mathbf{e}\|_2$;

(d) Assume the sequence $\{\mathbf{x}^l\}$ is generated by Algorithm 1, and $\|\mathbf{x}^l - \mathbf{x}^*\|_2 \geq c_3\|\mathbf{e}\|_2$ holds for any $1 \leq l \leq k$.

Then for any $1 \leq l \leq k$,

$$\|\mathbf{x}^l - \mathbf{x}^*\|_2^2 + \frac{2\delta c_2}{\tau c_3}\|\mathbf{x}^l - \mathbf{x}^*\|_2 \leq \|\mathbf{x}^{l-1} - \mathbf{x}^*\|_2^2.$$

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Simulation Results

General settings

- We choose

$$F(x) = (|x| - \rho x^2)\mathbf{1}_{|x| \leq 1/(2\rho)} + 1/(4\rho)\mathbf{1}_{|x| > 1/(2\rho)}.$$

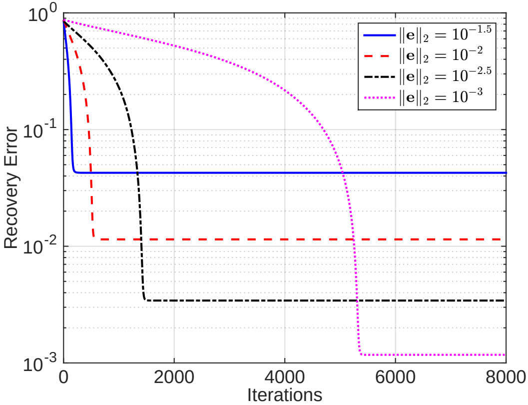
Then it can be calculated that when $\lambda < 1/(2\rho)$,

$$\mathrm{prox}_F(v_i, \lambda) = \frac{v_i - \lambda \mathrm{sign}(v_i)}{1 - 2\lambda\rho}\mathbf{1}_{\lambda < |v_i| \leq 1/(2\rho)} + v_i \mathbf{1}_{|v_i| > 1/(2\rho)}.$$

- We take random partial DCT measurements of a normalized sparse signal.

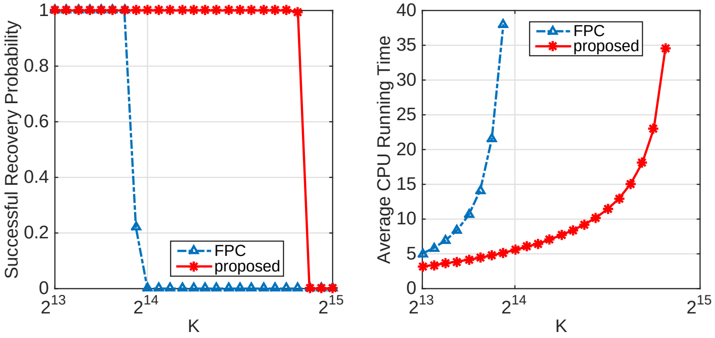
Experiment 1: Rate of Convergence

$$N = 2^{10},\ M = 2^8,\ K = 2^5,\ \rho = 5,\ \delta = 1, \text{ and } \tau = \sqrt{N}/\|\mathbf{e}\|_2$$



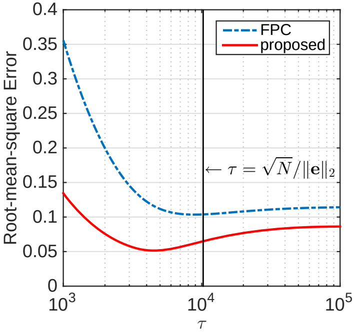
Experiment 2: Comparison in the Noiseless Case

$N = 2^{18}$, $M = 2^{16}$, K varies from 2^{13} to 2^{15}, $\rho = 120$, and $\tau = 10^6$



Experiment 3: Comparison in the Noisy Case

$N = 2^{18}$, $M = 2^{16}$, $K = 2^{13}$, $\rho = 120$, and (left) $\|\mathbf{e}\|_2 = 0.05$; (right) $\tau = \sqrt{N}/\|\mathbf{e}\|_2$



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Summary

- We propose a fast algorithm for the non-convex function regularized least squares problem.
- We prove that under some conditions, the iterates converge to a neighborhood of the sparse signal with superlinear convergence rate.
- Simulation results verify the theoretical results and show the superiority of the proposed algorithm compared with its convex counterpart.