# A SHORT-GRAPH FOURIER TRANSFORM VIA PERSONALIZED PAGERANK VECTORS

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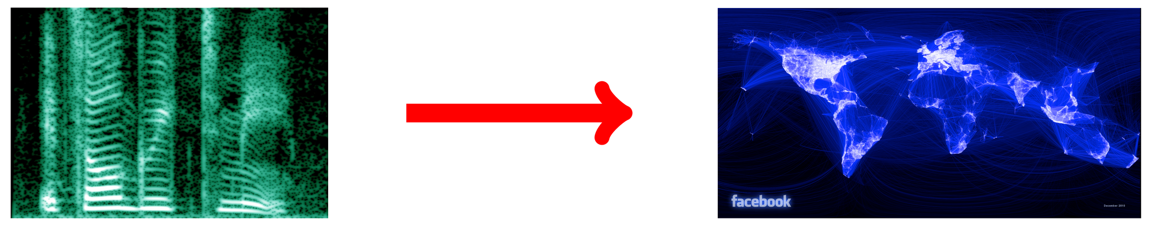



## Conceptual setting

- **New formulation** of the short-time Fourier transform (STFT) for **graph signals**
- Fundamental building block: personalized PageRank (PPR) vectors
- Connecting local spectral graph theory and localized spectral analysis of graph signals

## Local spectral analysis

The STFT as a three-steps algorithm:

1. translate the window $w$ by $u$
2. modulate the result by frequency $\xi$
3. take the convolution of the result with the signal $f$

Can be written as $\mathrm{STFT}_f = \langle f, \mathcal{M}_\xi T_u w \rangle$, where

- $(T_u g)(t) \doteq g(t-u)$ is the translation operator
- $(\mathcal{M}_\xi g)(t) \doteq g(t) e^{i\xi t}$ is the modulation operator

## Local spectral graph theory

Given $S \in V$, define the unit vector $\mathbf{s} = \mathrm{unit}(S)$ as

$$(\mathrm{unit}(S))_i \doteq \begin{cases} b / \mathrm{vol}\,(S) & \text{if } i \in S; \\ -b / \mathrm{vol}\,(V \setminus S) & \text{otherwise,} \end{cases}$$

where $b = \sqrt{\mathrm{vol}\,(S)\, \mathrm{vol}\,(V \setminus S) \,/\, \mathrm{vol}\,(V)}$.

Spectral decomposition with a twist:

$$\mathbf{x}^* = \operatorname*{argmin}_{\mathbf{x}} \mathbf{x}^{\mathrm{T}} \mathbf{L} \mathbf{x} \ \text{ s.t. } \begin{array}{l} \mathbf{x}^{\mathrm{T}} \mathbf{D} \mathbf{x} = 1, \\ \mathbf{x}^{\mathrm{T}} \mathbf{D} \mathbf{1} = 0, \\ \mathbf{x}^{\mathrm{T}} \mathbf{D} \mathbf{s} \geq \kappa. \end{array} \quad (1)$$

$\mathbf{x}^*$ is the solution to the PPR equation:

$$\mathbf{D}\mathbf{p} = (1-\alpha)(\mathbf{D}\mathbf{s}) + \alpha \mathbf{A}\mathbf{D}^{-1}(\mathbf{D}\mathbf{p}).$$

## Translation over graphs

We define the local window at node $i$ as

$$\mathbf{w}_i \doteq \max\,(0,\ \mathbf{x}_i^*) \,/\, \|\max\,(0,\ \mathbf{x}_i^*)\|_1 ,$$

where $\mathbf{x}_i^*$ the solution to Problem (1) with $\mathbf{s} = \mathrm{unit}(\{i\})$ and the maximum is taken entrywise.

## Modulation over graphs

For $k \in \{1, 2, \dots, n\}$, we define the graph modulation operator $M_k : \mathbb{R}^n \to \mathbb{R}^n$ by

$$M_k f \doteq \sqrt{\mathrm{vol}\,(V)}\ f \circ (\mathbf{D}^{-1/2}\mathbf{U})_{:k},$$

where $\mathbf{D}$ is the degree matrix, $\mathbf{U}$ are the eigenvectors of the normalized graph Laplacian, and $\circ$ is the entrywise multiplication.

## Short-graph Fourier transform

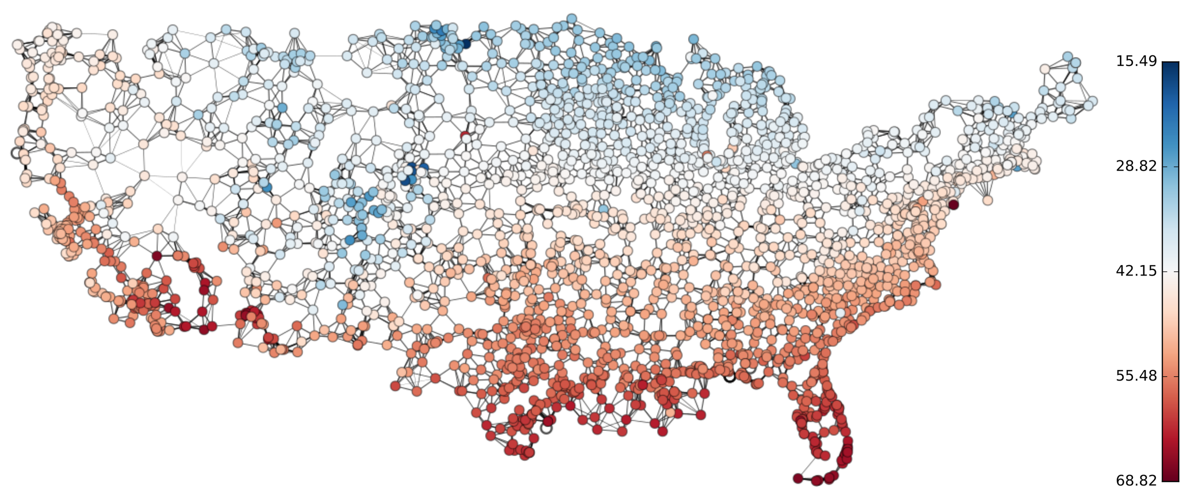
Let $\mathbf{f} \in \mathbb{R}^n$ be a signal over a graph $G = (V, E)$.
We define its short-graph Fourier transform at vertex $i \in V$ and frequency $k \in \{1, 2, \dots, n\}$ as

$$\mathrm{SGFT}_f(i, k) = \langle \mathbf{f}, M_k \mathbf{w}_i \rangle.$$

The spectrogram of $\mathbf{f}$ is defined as

$$\mathrm{spectrogram}_f(i, k) = |\,\mathrm{SGFT}_f(k, i)|^2.$$

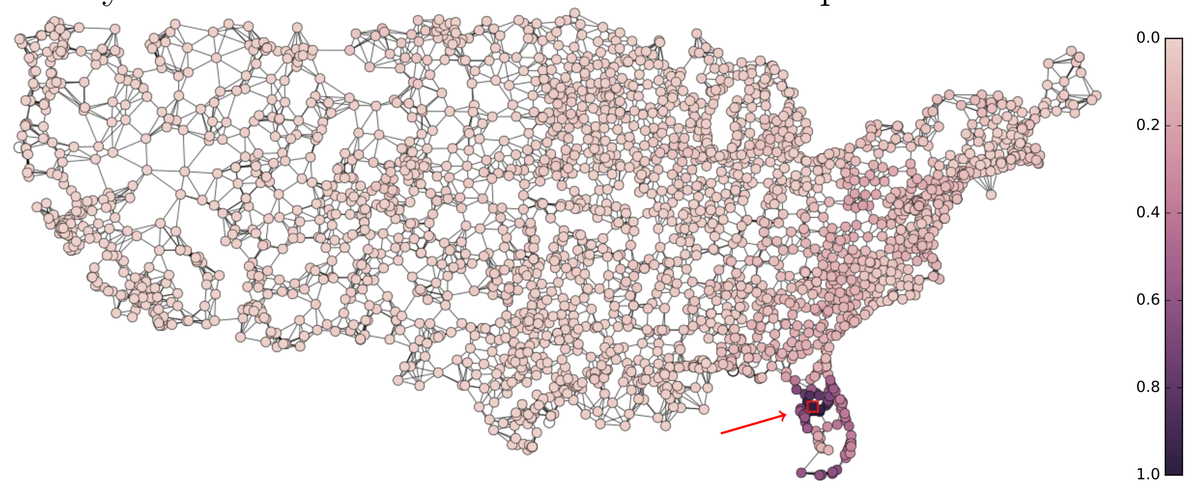
## Graph: Weather stations in the US




Average annual temperature in 2014
**Connectivity:** 6 spatial nearest neighbors
**Edge weight:** spatial distance between stations

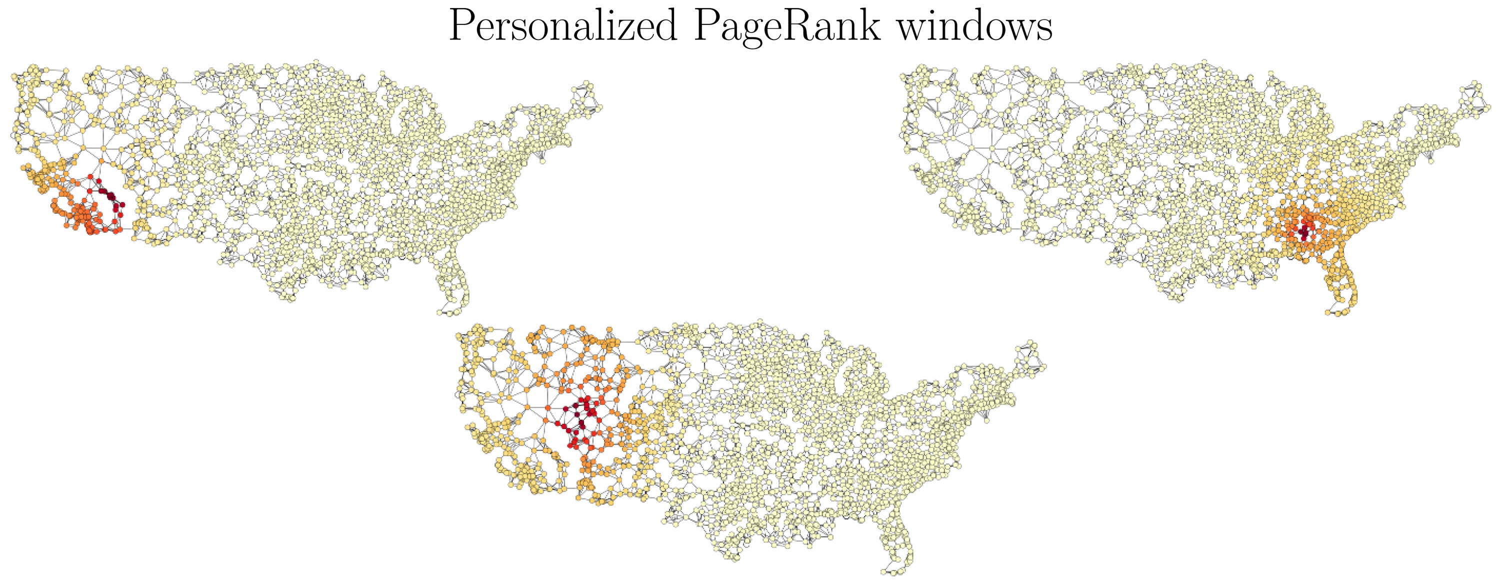
## Correlation of spectral signature

Analysis of the vertex marked with a red square & arrow

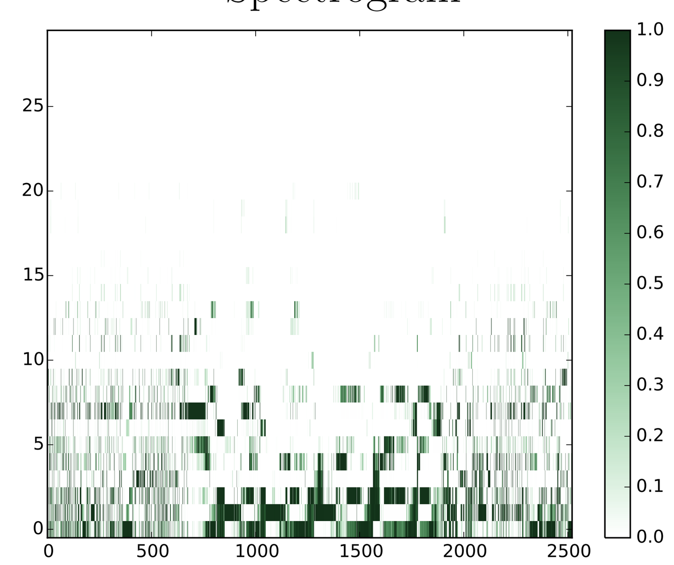



## Short-graph Fourier transform: Results

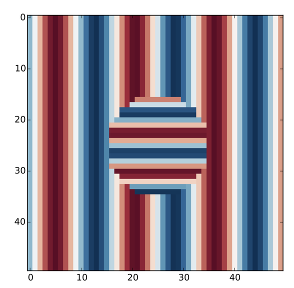
Personalized PageRank windows



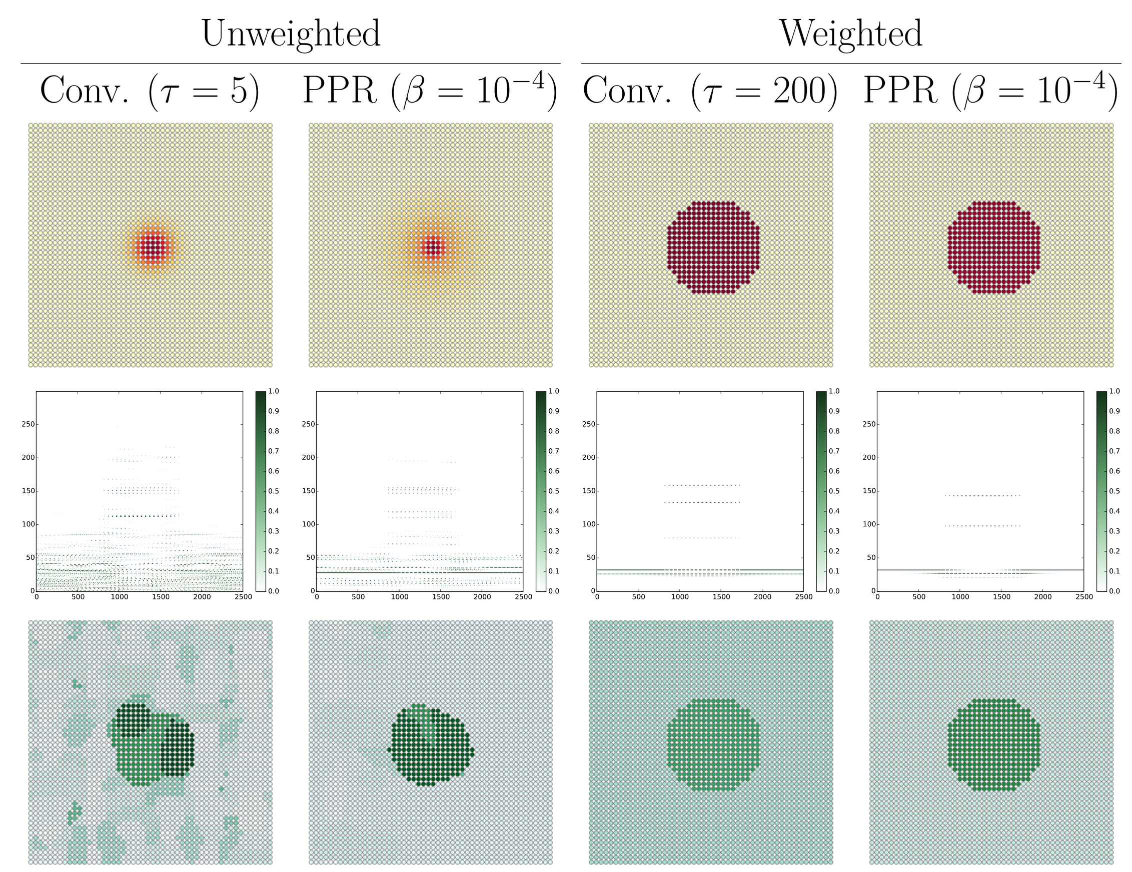
Spectrogram



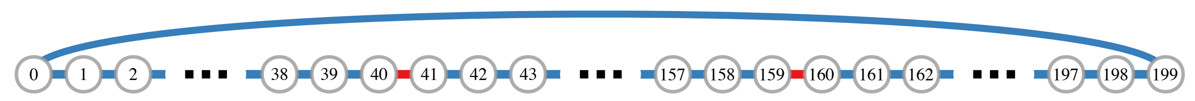

## Regular grid



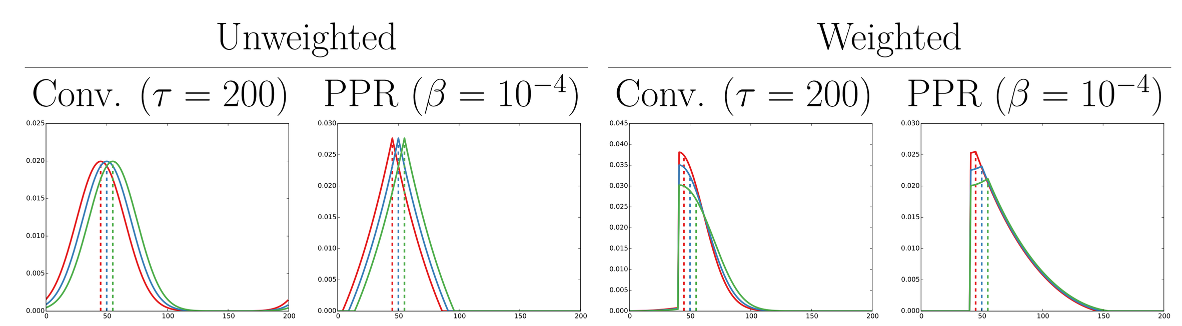
**Connectivity:** 4 neighborhood (periodicity in the edges)
**Weighted case:** edges connecting both waveforms are set to $10^{-5}$, all other edges $= 1$




## Window localization comparison




**Weighted case:** blue edges $= 1$, red edges $= 10^{-3}$




## Future work

Study theoretical properties

Extension to wavelets

## References

Mahoney, Orecchia, and Vishnoi, "A local spectral method for graphs: With applications to improving graph partitions and exploring data graphs locally," JMLR, 2012.
Shuman, Ricaud, and Vandergheynst, "A windowed graph Fourier transform," in SSP, 2012.

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