Review of Discrete-Time System

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Outline

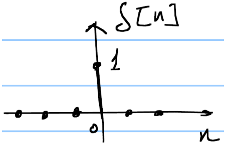
- Discrete-time signals: $\delta(n)$, $u(n)$, exponentials, sinusoids
- Transforms: ZT, FT
- Discrete-time system: LTI, causality, stability, FIR & IIR system
- Sampling of a continuous-time signal
- Discrete-time filters: magnitude response, linear phase
- Time-frequency relations: FS; FT; DTFT; DFT

Homework: Pick up a DSP text and review.

§0.1 Basic Discrete-Time Signals

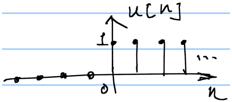
1. unit pulse (unit sample)

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$



2. unit step

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases}$$



Questions:

- What is the relation between $\delta[n]$ and $u[n]$?
- How to express any $x[n]$ using unit pulses?
  $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

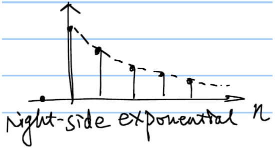
3. Sinusoids and complex exponentials

$$x_1[n] = A\cos(\omega_0 n + \theta)$$
$$x_2[n] = ae^{j\omega_0 n}$$

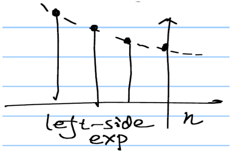
- $x_2[n]$ has real and imaginary parts; known as a single-frequency signal.

4. Exponentials

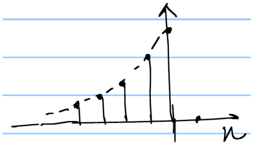
$x[n] = a^n u[n]$ $(0 < a < 1)$



$x[n] = a^n u[-n]$



$x[n] = a^{-n} u[-n]$



Questions:

Is $x_1[n]$ a single-frequency signal? Are $x_1[n]$ and $x_2[n]$ periodic?

§0.2 (1) Z-Transform

The Z-transform of a sequence $x[n]$ is defined as

$$\mathbb{X}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

In general, the region of convergence (ROC) takes the form of $R_1 < |z| < R_2$.

E.g.: $x[n] = a^n u[n]$: $\mathbb{X}(z) = \frac{1}{1-az^{-1}}$, ROC is $|z| > |a|$.

The same $\mathbb{X}(z)$ with a different ROC $|z| < |a|$ will be the ZT of a different $x[n] = -a^n u[-n-1]$.

The Fourier transform of a discrete-time signal $x[n]$

$$\mathbb{X}_{\mathrm{DTFT}}(\omega) = \mathbb{X}(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Often known as the Discrete-Time Fourier Transform (DTFT)
- If the ROC of $\mathbb{X}(z)$ includes the unit circle, we evaluate $\mathbb{X}(z)$ with $z = e^{j\omega}$, we call $\mathbb{X}(e^{j\omega})$ the Fourier Transform of $x[n]$
- The unit of frequency variable ω is radians
- $\mathbb{X}(\omega)$ is periodic with period 2π
- The inverse transform is $x[n] = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{X}(\omega)e^{j\omega n}d\omega$

§0.2 (2) Fourier Transform

Question: What is the FT of a single-frequency signal $e^{j\omega_0 n}$?

- Since the ZT of a^n does not converge anywhere except for $a = 0$, the FT for $x[n] = e^{j\omega_0 n}$ does not exist in the usual sense.
- But we can unite its FT as $2\pi\delta_a(\omega - \omega_0)$ for ω in the range between $0 < \omega < 2\pi$ and periodically repeating, by using a Dirac delta function $\delta_a(\cdot)$.

§0.2 (3) Parseval's Relation

Let $\mathbb{X}(\omega)$ and $\mathbb{Y}(\omega)$ be the FT of $x[n]$ and $y[n]$, then

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{X}(\omega)\mathbb{Y}^*(\omega)d\omega.$$

i.e., the inner product is preserved (except a multiplicative factor):

$$< x[n], y[n] >=< \mathbb{X}(\omega), \mathbb{Y}(\omega) > \cdot\frac{1}{2\pi}$$

1. If $x[n] = y[n]$, we have $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\mathbb{X}(\omega)|^2 d\omega$
2. Parseval's Relation suggests that the energy of $x[n]$ is conserved after FT and provides us two ways to express the energy.

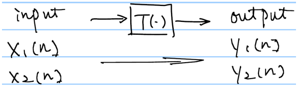
Question: Prove the Parseval's Relation.
(Hint: start with applying the definition of inverse DTFT for $x[n]$ to LHS)

§0.3 (1) Discrete-Time Systems

Question 1: How to characterize a general system?

§0.3 (2) Linear Time-Invariant Systems

Suppose



Linearity

$$\text{(input) } a_1 x_1[n] + a_2 x_2[n] \rightarrow \text{(output) } a_1 y_1[n] + a_2 y_2[n]$$

If the output in response to the input $a_1 x_1[n] + a_2 x_2[n]$ equals to $a_1 y_1[n] + a_2 y_2[n]$ for every pair of constants a_1 and a_2 and every possible $x_1[n]$ and $x_2[n]$, we say the system is linear.

Shift-Invariance (Time-Invariance)

$$\text{(input) } x_1[n-N] \rightarrow \text{(output) } y_1[n-N]$$

i.e., The output in response to the shifted input $x_1[n-N]$ equals to $y_1[n-N]$ for all integers N and all possible $x_1[n]$.

§0.3 (3) Impulse Response of LTI Systems

An LTI system is both linear and shift-invariant. Such a system can be completely characterized by its impulse response $h[n]$:

$$\text{(input) } \delta[n] \rightarrow \text{(output) } h[n]$$

Recall all $x[n]$ can be represented as $x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$
$\Rightarrow$ By LTI property:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

§0.3 (4) Input-Output Relation of LTI Systems

The input-output relation of an LTI system is given by a convolution summation:

$$\underbrace{y[n]}_{\text{output}} = h[n] * \underbrace{x[n]}_{\text{input}} = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

- The transfer-domain representation is $\mathbb{Y}(z) = H(z)\mathbb{X}(z)$, where

$$H(z) = \frac{\mathbb{Y}(z)}{\mathbb{X}(z)} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

is called the <u>transfer function</u> of the LTI system.

§0.3 (5) Rational Transfer Function

A major class of transfer functions we are interested in is the rational transfer function:

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{m=0}^{N} a_m z^{-m}}$$

- $\{a_n\}$ and $\{b_n\}$ are finite and possibly complex.
- N is the order of the system if $B(z)/A(z)$ is irreducible.

§0.3 (6) Causality

The output doesn't depend on future values of the input sequence.
(important for processing a data stream in real-time with low delay)

An LTI system is causal iff $h[n] = 0 \ \forall \ n < 0$.

Question: What property does $H(z)$ have for a causal system?

Pitfalls: note the spelling of words "casual" vs. "causal".

§0.3 (7) FIR and IIR systems

- A causal N-th order finite impulse response (FIR) system can have its transfer function written as $H(z) = \sum_{n=0}^{N} h[n] z^{-n}$

- A causal LTI system that is not FIR is said to be IIR (infinite impulse response).

  e.g. exponential signal $h[n] = a^n u[n]$:
  its corresponding $H(z) = \frac{1}{1 - az^{-1}}$.

§0.3 (8) Stability in the BIBO sense

BIBO: bounded-input bounded-output

An LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

i.e. its impulse response is absolutely summable.

This sufficient and necessary condition means that ROC of $H(z)$ includes unit circle: $\because |H(z)|_{z=e^{j\omega}} \leq \sum_n |h[n]| \times 1 < \infty$

If $H(z)$ is rational and $h[n]$ is causal (s.t. ROC takes the form $|z| > r$), the system is stable **iff** all poles are inside the unit circle (such that the ROC includes the unit circle).

§0.4 (1) Fourier Transform

We use the subscript "a" to denote continuous-time (analog) signal and drop the subscript if the context is clear.

The Fourier Transform of a continuous-time signal $x_a(t)$

$$\begin{cases} \mathbb{X}_a(\Omega) \triangleq \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt & \text{"projection"} \\ x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbb{X}_a(\Omega) e^{j\Omega t} d\Omega & \text{"reconstruction"} \end{cases}$$

- $\Omega = 2\pi f$ and is in radian per second
- f is in Hz (i.e., cycles per second)

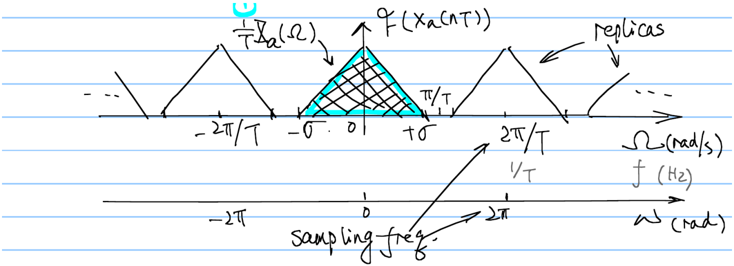
§0.4 (2) Sampling

Consider a sampled signal $x[n] \triangleq x_a(nT)$.

- $T > 0$: sampling period; $2\pi/T$: sampling (radian) frequency

The Discrete Time Fourier Transform of $x[n]$ and the Fourier Transform of $x_a(t)$ have the following relation:

$$\mathbb{X}(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathbb{X}_a(\Omega - \frac{2\pi k}{T})|_{\Omega=\frac{\omega}{T}}$$

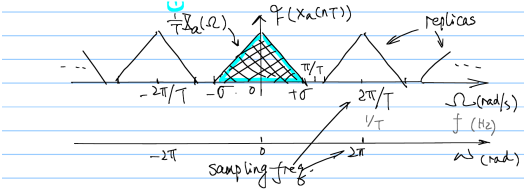


§0.4 (3) Aliasing

- If $\mathbb{X}_a(\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ (i.e., band limited), there is no overlap between $\mathbb{X}_a(\Omega)$ and its shifted replicas.

  Can recover $x_a(t)$ from the sampled version $x[n]$ by retaining only one copy of $\mathbb{X}_a(\Omega)$. This can be accomplished by interpolation/filtering.

- Otherwise, overlap occurs. This is called aliasing.

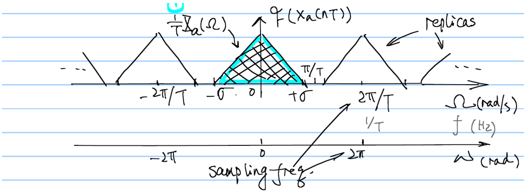


Reference: Chapter 7 “Sampling” in Oppenheim et al. Signals and Systems Book

- Let $x_a(t)$ be a band-limited signal with $\mathbb{X}_a(\Omega) = 0$ for $|\Omega| \geq \sigma$, then $x_a(t)$ is uniquely determined by its samples $x_a(nT)$, $n \in \mathbb{Z}$,

> if the sampling frequency $\Omega_s \triangleq 2\pi/T$ satisfies $\Omega_s \geq 2\sigma$.

- In the ω domain, 2π is the (normalized) sampling rate for any sampling period T.
  Thus the signal bandwidth can at most be π to avoid aliasing.



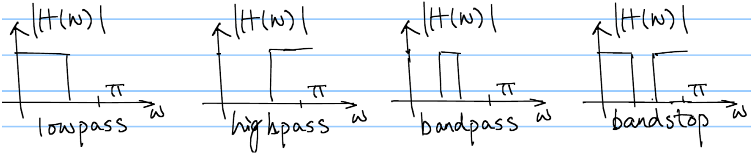
§0.5 Discrete-Time Filters

1. A Digital Filter is an LTI system with rational transfer function. The frequency response $H(e^{j\omega})$ specifies the properties of a filter:

$$H(\omega) = |H(\omega)|e^{j\phi(\omega)}$$

$|H(\omega)|$: magnitude response
$\phi(\omega)$: phase response

2. **Magnitude response** determines the type of filters:



3. **Linear-phase filter**: phase response $\phi(\omega)$ is linear in ω.
   - Linear phase is usually the minimal phase distortion we can expect.
   - A real-valued linear-phase FIR filter of length N normally is either symmetric $h[n] = h[N - n]$ or anti-symmetric $h[n] = -h[N - n]$.

§0.6 Relations of Several Transforms (answer)

TRANSFORM	*TIME-DOMAIN* (*Analysis*)	*FREQUENCY-DOMAIN* (*Synthesis*)
Fourier Series (*FS*)		
Fourier Transform (*FT*)		
Discrete-Time Fourier Transform (*DTFT*)		
Discrete Fourier Transform (*DFT*)		

§0.6 Relations of Several Transforms

TRANSFORM	*TIME-DOMAIN* *(Analysis)*	*FREQUENCY-DOMAIN* *(Synthesis)*
Fourier Series (*FS*)	$x(t)$ continuous periodic $X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-j2\pi nt/T}dt$	X_n discrete aperiodic $x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi nt/T}$
Fourier Transform (*FT*)	$x(t)$ continuous aperiodic $X(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$ (or in f where $\Omega = 2\pi f$)	$X(\Omega)$ continuous aperiodic $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)e^{j\Omega t}d\Omega$
Discrete-Time Fourier Transform (*DTFT*)	$x[n]$ discrete aperiodic $X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$	$X(\omega)$ continuous periodic $x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(\omega)e^{j\omega n}d\omega$
Discrete Fourier Transform (*DFT*)	$x[n]$ discrete periodic $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$ (where $W_N^{kn} = e^{-j2\pi kn/N}$)	$X[k]$ discrete periodic $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$

§0.3 (1) Discrete-Time Systems

Question 1: How to characterize a general system?

Ans: by its input-output response (which may require us to enumerate all possible inputs, and observe and record the corresponding outputs)

Question 2: Why are we interested in LTI systems?

Ans: They can be completely characterized by just one response - the response to impulse input