Consistent Estimation of Randomly Sampled Ornstein-Uhlenbeck Process Long-Run Mean for Long-Term Target State Prediction

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Problem

Ship traffic monitoring is a foundation for many maritime applications. At the sime time, vessels in the open sea are seldom continuously observed, and the problem of long-term target state prediction becomes crucial. Unfortunately, the issue has been overlooked in the target tracking literature and traditional motion models, such as the Nearly-Constant Velocity (NCV) model, are not suitable for long-term predictions. A novel target motion model, based on the stochastic, mean-reverting Ornstein-Uhlenbeck process was recently proposed and validated [1, 2] to properly characterize the motion of non-maneuvering ships at sea and validated against a very large real-world data set.

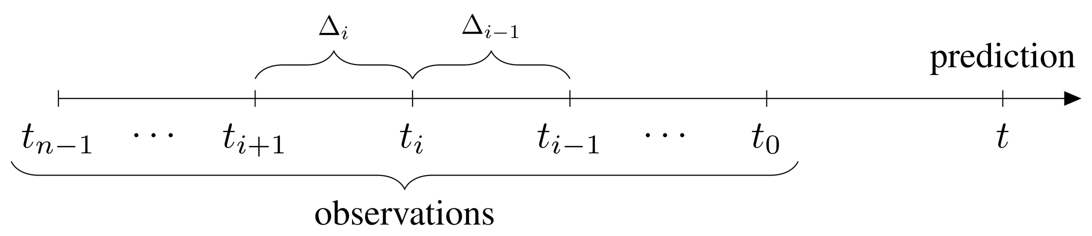
Contributions

Unlike NCV, which has only one parameter, the OU process is governed by a set of parameters: the long-run mean $\boldsymbol{v}$, the reversion rate $\boldsymbol{\Gamma}$, and the process noise covariance $\boldsymbol{C}$, which are clearly unknown in real applications.

At the same time, knowledge of these parameters is required in order to perform the optimal prediction in the Bayesian sense. This paper focuses on the lack of knowledge of the long-run mean $\boldsymbol{v}$ and shows how the target state prediction is affected by it in the long term. The main contribution of this work is a closed form for the error covariance matrix of the SME of $\boldsymbol{v}$, showing that:

- the SME is $\sqrt{n}$-consistent with a random sampling interval, and
- it reaches the CRLB when the sampling time is constant.

Prediction



Optimal prediction

$$\mathrm{E}\left[\boldsymbol{s}(t)\,\middle|\,\boldsymbol{s}(t_0),\boldsymbol{v}\right] = \widetilde{\boldsymbol{R}}\boldsymbol{\Phi}(t-t_0,\boldsymbol{\gamma})\,\widetilde{\boldsymbol{R}}^{-1}\boldsymbol{s}(t_0) + \widetilde{\boldsymbol{R}}\boldsymbol{\Psi}(t-t_0,\boldsymbol{\gamma})\,\boldsymbol{R}^{-1}\boldsymbol{v}$$

$$\boldsymbol{\Phi}(t,\boldsymbol{\gamma}) = \begin{bmatrix} \boldsymbol{I} & \left(\boldsymbol{I}-\mathrm{e}^{-\boldsymbol{\Gamma}t}\right)\boldsymbol{\Gamma}^{-1} \\ \boldsymbol{0} & \mathrm{e}^{-\boldsymbol{\Gamma}t} \end{bmatrix}$$ state transition matrix

$$\boldsymbol{\Psi}(t,\boldsymbol{\gamma}) = \begin{bmatrix} t\,\boldsymbol{I}-\left(\boldsymbol{I}-\mathrm{e}^{-\boldsymbol{\Gamma}t}\right)\boldsymbol{\Gamma}^{-1} \\ \boldsymbol{I}-\mathrm{e}^{-\boldsymbol{\Gamma}t} \end{bmatrix}$$ control input function

Knowledge of the process parameters is required for optimal prediction that, interestingly, is independent of the true long-run mean value. The overall prediction error covariance can be seen as the sum of the ideal ($\boldsymbol{v}_n = \boldsymbol{v}$) prediction error covariance $\boldsymbol{C}^*(t-t_0)$ and an additional term $\boldsymbol{C}_n$ that accounts for the imperfect knowledge of $\boldsymbol{v}$.

Prediction covariance

$$\boldsymbol{C}_n(t-t_0) = \boldsymbol{C}^*(t-t_0) + \widetilde{\boldsymbol{R}}\boldsymbol{\Psi}(t-t_0,\boldsymbol{\gamma})\,\boldsymbol{C}_n\boldsymbol{\Psi}(t-t_0,\boldsymbol{\gamma})^{\mathsf{T}}\,\widetilde{\boldsymbol{R}}^{\mathsf{T}}$$

Model

Ornstein-Uhlenbeck (OU) model

$$\mathrm{d}\boldsymbol{s}(t) = \boldsymbol{A}\,\boldsymbol{s}(t)\,\mathrm{d}t + \boldsymbol{G}\,\boldsymbol{v}\,\mathrm{d}t + \boldsymbol{B}\;\mathrm{d}\boldsymbol{w}(t)$$

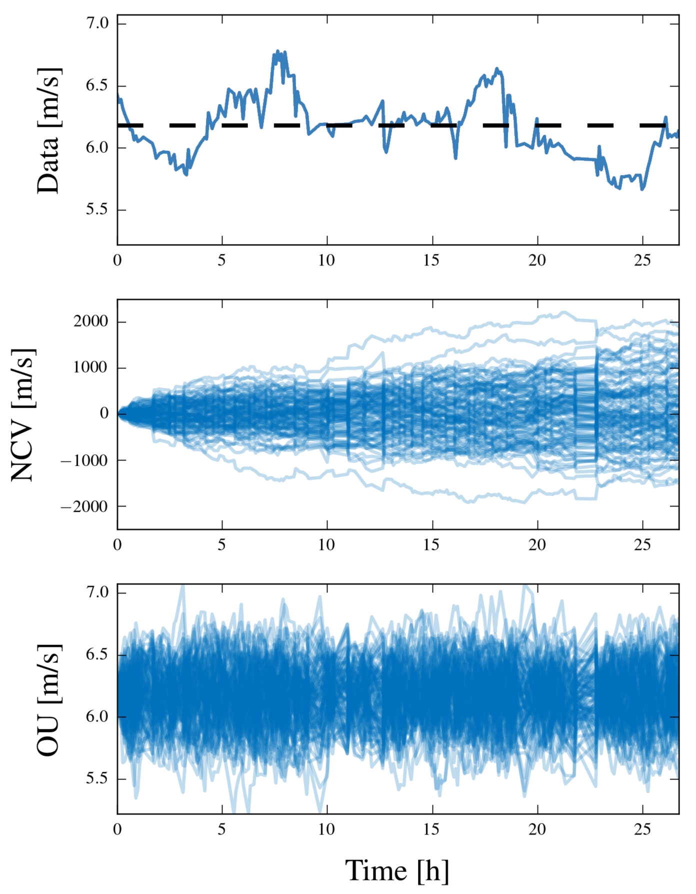
$$\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I} \\ 0 & \boldsymbol{R}\boldsymbol{\Gamma}\boldsymbol{R}^{-1} \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 0 \\ \boldsymbol{C} \end{bmatrix},\ \boldsymbol{G} = \begin{bmatrix} 0 \\ \boldsymbol{R}\boldsymbol{\Gamma}\boldsymbol{R}^{-1} \end{bmatrix}$$

The NCV target motion equation has a similar form and, interestingly, can be seen as a *special* OU process, with a null reversion rate matrix $\boldsymbol{\Gamma} = \boldsymbol{0}$, meaning that there is not any tendency of the walk to move back towards the long-run mean.

Nearly Constant Velocity (NCV) model

$$\mathrm{d}\boldsymbol{s}(t) = \boldsymbol{A}\,\boldsymbol{s}(t)\,\mathrm{d}t + \boldsymbol{B}\;\mathrm{d}\boldsymbol{w}(t)$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I} \\ 0 & 0 \end{bmatrix},\quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \boldsymbol{C} \end{bmatrix}$$



Actual speed of a vessel under way

Synthetic realizations of the NCV process (*same initial state*)

Synthetic realizations of the OU process (*same initial state*)

Long-run mean estimation

Sample Mean Estimator (SME) $\quad \boldsymbol{v}_n = \frac{1}{n}\sum_{i=0}^{n-1}\dot{\boldsymbol{x}}_i \quad$ independent of the other process parameters

The SME error covariance can be computed by conditioning on the observation time intervals $\mathcal{D}_n = \{\Delta_i\}_{i=1}^n$. Interestingly, the SME error covariance has a closed form also for random i.i.d. Δ_i's.

SME error covariance

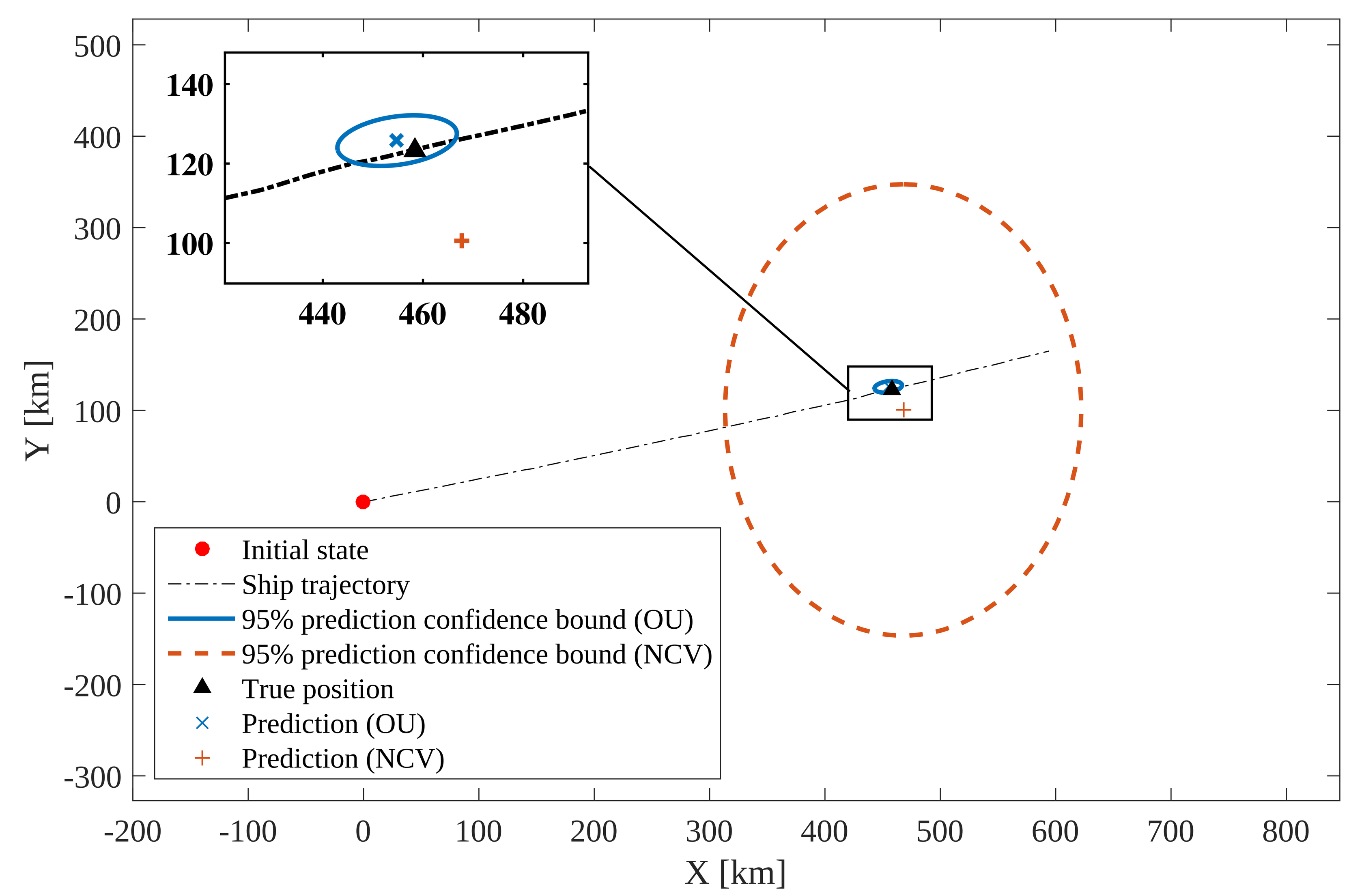
$$\boldsymbol{C}_n = \boldsymbol{\Sigma}_\infty \circ \begin{bmatrix} a_n^{(\Delta)}(\gamma_x) & b_n^{(\Delta)}(\gamma_x,\gamma_y) \\ b_n^{(\Delta)}(\gamma_x,\gamma_y) & a_n^{(\Delta)}(\gamma_y) \end{bmatrix}$$

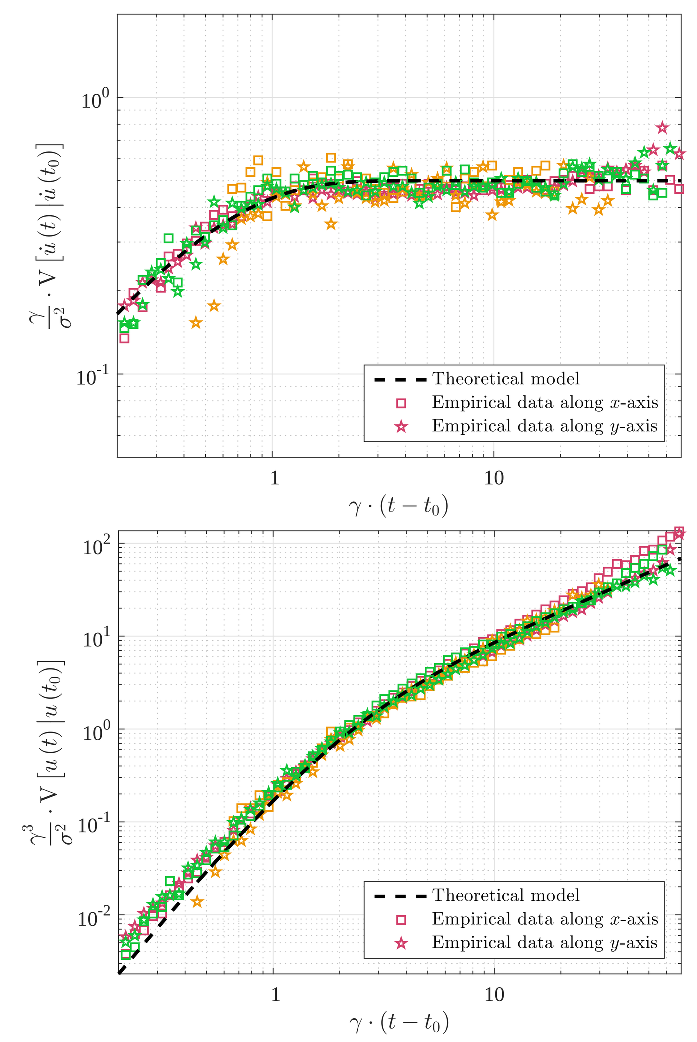
$$a_n^{(\Delta)}(\delta) = \alpha_n\left(\kappa_\Delta(\gamma_x)\right)$$

$$b_n^{(\Delta)}(\delta_1,\delta_2) = \beta_n\left(\kappa_\Delta(\gamma_x),\kappa_\Delta(\gamma_y)\right)$$

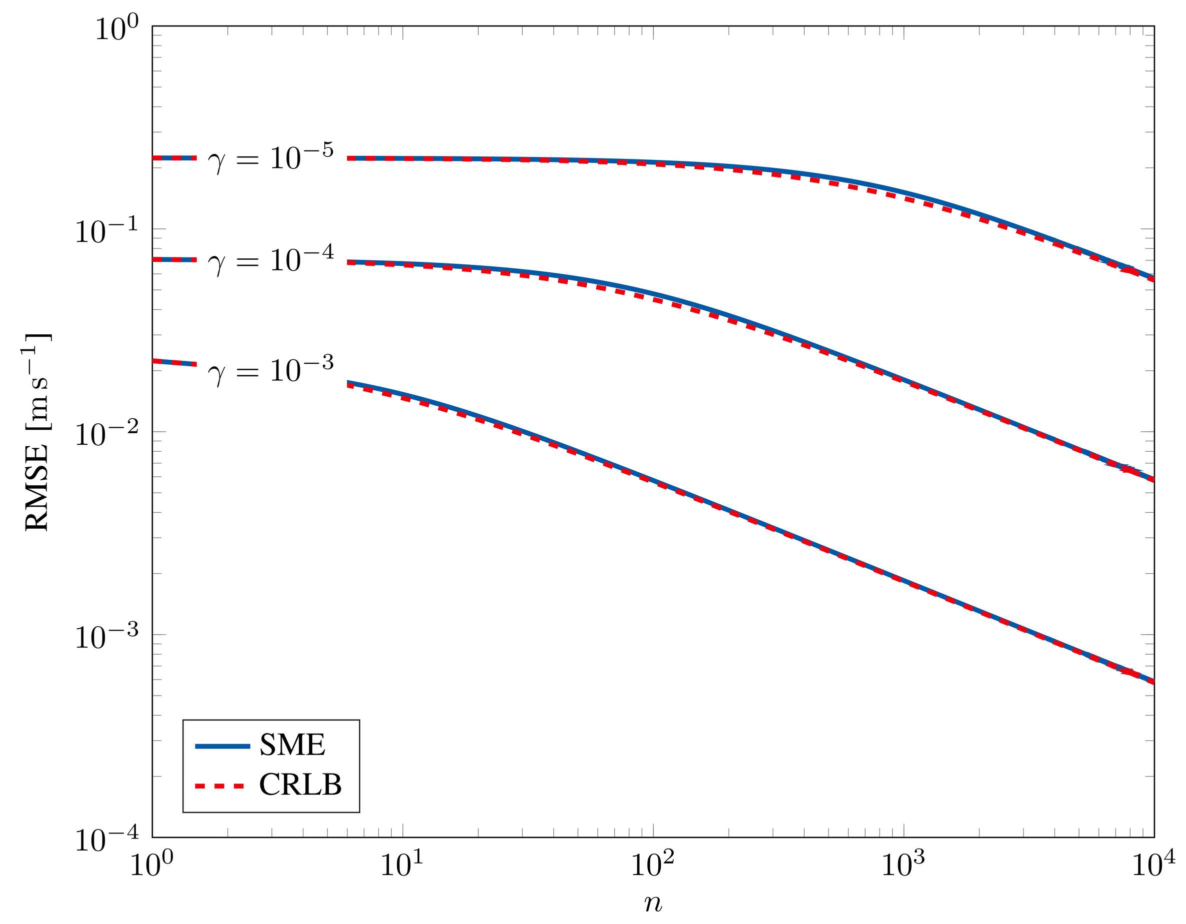
$$\kappa_\Delta(\gamma) = -\log \mathrm{E}\left[\mathrm{e}^{-\gamma\Delta}\right]$$

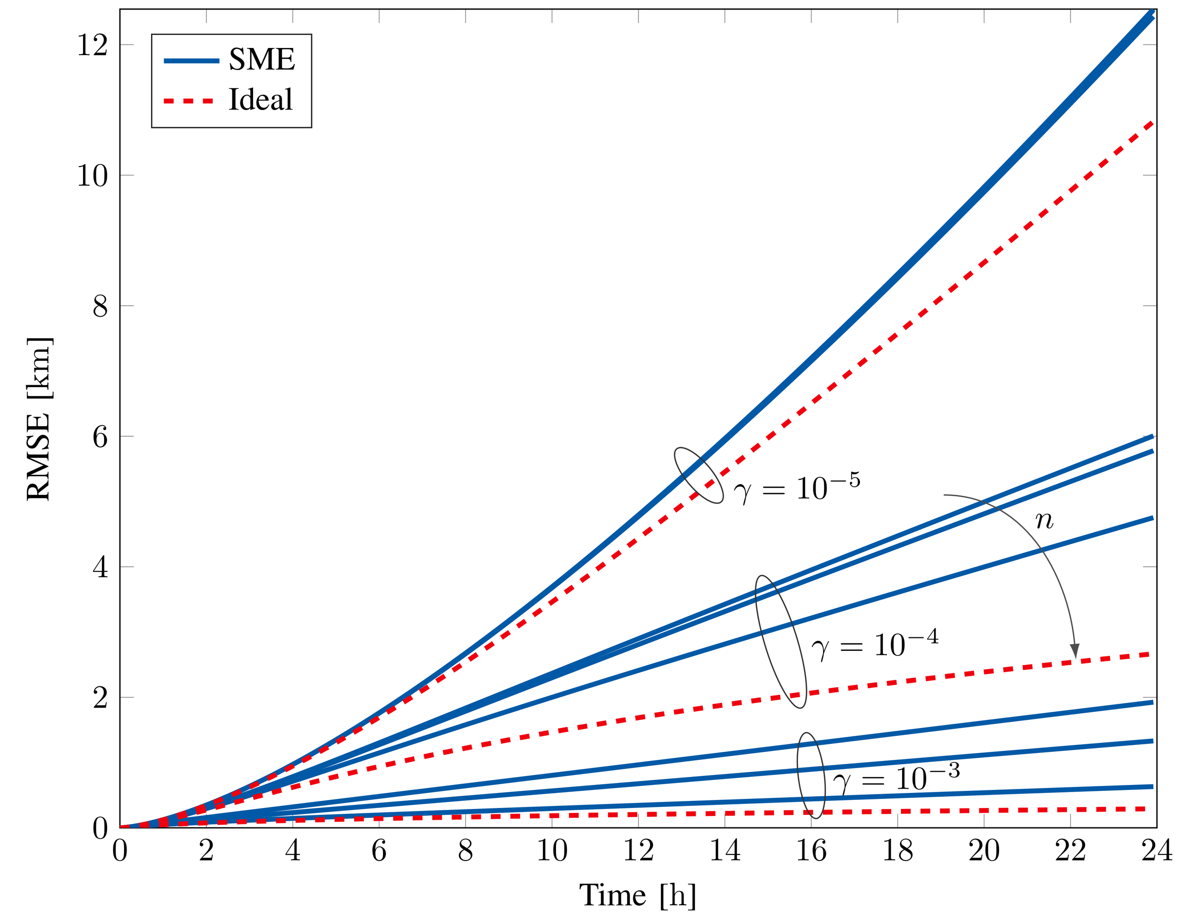
Validation (real-world dataset, ≈200,000 contacts)





Results





References

[1] L. M. Millefiori, P. Braca, K. Bryan, and P. Willett, "Modeling vessel kinematics using a stochastic mean-reverting process for long-term prediction," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 52, no. 5, pp. 2313–2330, October 2016.

[2] ——, "Long-term vessel kinematics prediction exploiting mean-reverting processes," in *2016 19th International Conference on Information Fusion (FUSION)*, July 2016, pp. 232–239.