New Results on the Sum of Two Generalized Gaussian Random Variables

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Motivation

- ✓ Gaussian noise is widely used in communication systems
- ✓ In sensor networks and local spectrum sensing, the deployed noise is Generalized Gaussian (GG)
- ✓ In UWB, interference and noise can be modeled as GG distribution
- ✓ Noise + Interference can be modeled as sum of GG random variables
- ✓ The statistics of the SGG distribution (PDF, CDF, moments, cumulant...) should be studied
- ✓ An approximation of the SGG by a single GG random can be used to get simplified results

Definition

- ✓ GG distribution is characterized by a parameter α, called shaping parameter (or exponent parameter)
- ✓ The PDF of GGD(μ, σ) is given as

$$f_X(x) = \frac{\alpha\Lambda}{2\Gamma(1/\alpha)} \exp\left(-\Lambda^{\alpha}|x-\mu|^{\alpha}\right) \ \forall x \in \mathbb{R},$$

where $\Lambda = \frac{\Lambda_0}{\sigma} = \frac{1}{\sigma}\sqrt{\frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)}}$, a normalization coefficient.

- ✓ For $\alpha = 1, 2, \infty$, we get Laplacian, Gaussian, and uniform distributions, respectively.

Cumulative Distribution Function

- ✓ The complementary CDF is obtained as

$$Q_\alpha(x) = \frac{1}{2\Gamma(1/\alpha)}\Gamma(1/\alpha, \Lambda_0^\alpha x^\alpha), \quad \text{for x} \geq 0,$$

- ✓ The CDF is given by

$$F_X(x) = 1 - Q_\alpha\left(\frac{x-\mu}{\sigma}\right).$$

- ✓ Q_α is reduced to the classical Gaussian Q-function for $\alpha = 2$.

Characteristic Function

- ✓ CHF is the Fourier Transform of the PDF
- ✓ PDF is even ($\mu = 0$) $\Rightarrow$ CHF becomes cosine transform of the PDF

$$\varphi_\alpha(t) = \int_0^\infty \cos(tx) f_X(x) dx$$

- ✓ Using alternative expressions of cos and f_X, CHF appears as an integral of 2 Fox H-functions
- ✓ Using integral identity[1], the CHF is expressed as

$$\varphi_\alpha(t) = \frac{\sqrt{\pi}}{\Gamma(1/\alpha)} \mathrm{H}_{1,2}^{1,1}\left[\frac{\sigma^2 \Gamma(1/\alpha)}{4\Gamma(3/\alpha)} t^2 \middle| \begin{matrix} (1-\frac{1}{\alpha}, \frac{2}{\alpha}) \\ (0,1), (\frac{1}{2}, 1) \end{matrix} \right]$$

[1] A. Kilbas and M. Saigo. *H-Transforms : Theory and Applications (Analytical Method and Special Function)*. 1st edition. CRC Press, 2004.

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MGF, Moment, and Cumulant

- ✓ The MGF is obtained from the CHF as $M_\alpha(t) = \varphi_\alpha(-it)$
- ✓ The odd moments are equal to 0, and the even moments are

$$m_{2n}(X) = \mathbb{E}[X^{2n}] = \sigma^{2n} \frac{\Gamma(1/\alpha)^n}{\Gamma(3/\alpha)^n} \frac{\Gamma(\frac{2n+1}{\alpha})}{\Gamma(1/\alpha)}$$

- ✓ Cumulant generating function: $K_\alpha(t) = \log M_\alpha(t)$
- ✓ New results: the even cumulants of zero mean GGD

$$k_{2n}(X) = -\sum_{m_1+2m_2+\ldots nm_n=n} \frac{(2n)!(m_1+\cdots+m_n-1)!}{m_1!m_2!\ldots m_n!} \prod_{1\leq j\leq n} \left(-\frac{\sigma^{2j}\Gamma(1/\alpha)^j\Gamma(\frac{2j+1}{\alpha})}{\Gamma(3/\alpha)^j\Gamma(1/\alpha)(2j)!}\right)^{m_j}$$

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Kurtosis

✓ Measures the tailedness of the distribution (i.e. more kurtosis means heavier tail)

✓ For the GGD, the kurtosis expressed as

$$Kurt(X) = \frac{k_4(X)}{k_2(X)^2} = \frac{\Gamma(1/\alpha)\Gamma(5/\alpha)}{\Gamma(3/\alpha)^2} - 3$$

✓ Examples:

- $Kurt(Laplace) = 3$, heavy tail
- $Kurt(Gaussian) = 0$, normal tail
- $Kurt(Uniform) = -1.2$, no positive-valued tail

✓ Applications: estimate the distribution parameter...

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Characteristic Function of the Sum

- ✓ CHF of the sum of 2 independent GGD is the product of their CHFs $\Rightarrow \varphi_Z(t) = \varphi_X(t)\varphi_Y(t)$
- ✓ The PDF of the sum, $f_Z(z)$, is the inverse cosine transform of the CHF

$$f_Z(z) = \frac{1}{2\Gamma(1/\alpha)\Gamma(1/\beta)} \int_0^\infty \cos\left(t(\mu - z)\right) \mathrm{H}_{1,2}^{1,1}\left[A\ t^2 \,\middle|\, \begin{matrix} (1-\frac{1}{\alpha}, \frac{2}{\alpha}) \\ (0,1), (\frac{1}{2}, 1) \end{matrix}\right] \times \mathrm{H}_{1,2}^{1,1}\left[B\ t^2 \,\middle|\, \begin{matrix} (1-\frac{1}{\beta}, \frac{2}{\beta}) \\ (0,1), (\frac{1}{2}, 1) \end{matrix}\right] dt,$$

where $\mu = \mu_1 + \mu_2$, $A = \frac{\sigma_1^2\Gamma(1/\alpha)}{4\Gamma(3/\alpha)}$, and $B = \frac{\sigma_2^2\Gamma(1/\beta)}{4\Gamma(3/\beta)}$

- ✓ The CDF is the primitive of the previous formula that vanishes at $-\infty$, so it is the inverse sine transform of $\varphi_Z(t)$.

PDF and CDF of the Sum

✓ PDF of the sum in terms of the BFHF

$$f_Z(z) = \frac{\sqrt{\pi}}{\Gamma(1/\alpha)\Gamma(1/\beta)|z-\mu|} \times \mathrm{H}^{0,1;1,1;1,1}_{2,0;1,2;1,2}\left[\frac{4A}{(z-\mu)^2}, \frac{4B}{(z-\mu)^2}\,\middle|\, \begin{matrix}(\frac{1}{2},1,1),(0,1,1)\\ \text{———}\end{matrix} \,\middle|\, \begin{matrix}(1-\frac{1}{\alpha},\frac{2}{\alpha})\\ (0,1),(\frac{1}{2},1)\end{matrix} \,\middle|\, \begin{matrix}(1-\frac{1}{\beta},\frac{2}{\beta})\\ (0,1),(\frac{1}{2},1)\end{matrix}\right] \quad (1)$$

✓ CDF of the sum

$$F_Z(z) = \frac{1}{2} + \frac{\sqrt{\pi}\; sign(z-\mu)}{2\Gamma(1/\alpha)\Gamma(1/\beta)} \times \mathrm{H}^{0,1;1,1;1,1}_{2,0;1,2;1,2}\left[\frac{4A}{(z-\mu)^2}, \frac{4B}{(z-\mu)^2}\,\middle|\, \begin{matrix}(\frac{1}{2},1,1),(1,1,1)\\ \text{———}\end{matrix} \,\middle|\, \begin{matrix}(1-\frac{1}{\alpha},\frac{2}{\alpha})\\ (0,1),(\frac{1}{2},1)\end{matrix} \,\middle|\, \begin{matrix}(1-\frac{1}{\beta},\frac{2}{\beta})\\ (0,1),(\frac{1}{2},1)\end{matrix}\right] \quad (2)$$

Statistics of the Sum

- ✓ MGF : $M_Z(t) = \varphi_Z(-it)$
- ✓ Moment: $m_{2n+1}(Z) = 0$ and

$$m_{2n}(Z) = \frac{\sigma_2^{2n}\Gamma(\frac{1}{\beta})^n}{\Gamma(\frac{1}{\alpha})\Gamma(\frac{1}{\beta})\Gamma(\frac{3}{\beta})^n}\sum_{k=0}^{n}\binom{2n}{2k}\left(\frac{\sigma_1^2}{\sigma_2^2}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{3}{\beta})}{\Gamma(\frac{3}{\alpha})\Gamma(\frac{1}{\beta})}\right)^k\Gamma\left(\frac{2k+1}{\alpha}\right)\Gamma\left(\frac{2n-2k+1}{\beta}\right)$$

- ✓ Kurtosis:

$$Kurt(Z) = \frac{\sigma_1^4}{\sigma^4}\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{5}{\alpha})}{\Gamma(\frac{3}{\alpha})^2} + \frac{\sigma_2^4}{\sigma^4}\frac{\Gamma(\frac{1}{\beta})\Gamma(\frac{5}{\beta})}{\Gamma(\frac{3}{\beta})^2} + 6\frac{\sigma_1^2\sigma_2^2}{\sigma^4} - 3,$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2$

Motivation

- ✓ Similar distribution properties of the sum of 2 GGD and one GGD[2]
- ✓ Approximation of the sum distribution by one GGD
- ✓ Estimation of one shape parameter instead of two
- ✓ Reducing the complexity of the expression of the PDF of the sum (BFHF)
- ✓ Simplification of the PDF/CDF expressions
- ✓ Less complexity expressions of system performance metrics (BER, SER...)

[2] Qian Zhao, Hong-Wei Li, and Yuan-Tong Shen. "On the sum of generalized Gaussian random signals". In: *Proc. of the 7th International Conference on Signal Processing (ICSP'2004)*. Vol. 1. Beijing, China, 2004, 50–53 vol.1.

Kurtosis Approach

- ✓ Equalize the Kurtosis of the sum to the Kurtosis of the new distribution (Z_γ) with shape parameter γ

$$Kurt(Z_\gamma) = Kurt(Z)$$

$$\Longleftrightarrow$$

$$\frac{\Gamma(\frac{1}{\gamma})\Gamma(\frac{5}{\gamma})}{\Gamma(\frac{3}{\gamma})^2} = \frac{1}{(1+\delta)^2}\left(\delta^2\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{5}{\alpha})}{\Gamma(\frac{3}{\alpha})^2} + \frac{\Gamma(\frac{1}{\beta})\Gamma(\frac{5}{\beta})}{\Gamma(\frac{3}{\beta})^2} + 6\delta\right),$$

where $\delta = \frac{\sigma_1^2}{\sigma_2^2}$

- ✓ Solve the system $h(\gamma) = C$, with unknown γ and C is positive known value

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Kurtosis Approach (2)

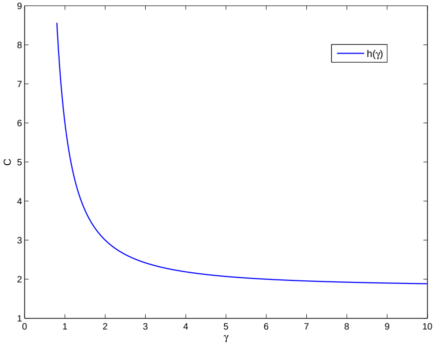

Figure 1: The curve of $h(\gamma)$ for positive values of γ.

✓ $h(\cdot)$ is bijection (Fig.1) $\Rightarrow$ the equation $h(\gamma) = C$ has unique solution γ_{Kurt} (examples of γ_{Kurt} in Table I)

Best Tail & CDF Approximation methods

- ✓ Best Tail: Minimizing the square error of the desired tail (defined for $z \geq n\sigma$)

$$\gamma_{Tail} = \arg\min_{\gamma > 0} \int_{n\sigma}^{\infty} \left(f_\gamma(z) - f_Z(z)\right)^2 dz$$

- ✓ CDF Approximation: Minimize the error between the exact and approximated CDF

$$\gamma_{CDF} = \arg\min_{\gamma > 0} \int_{0}^{\infty} \left(F_\gamma(z) - F_Z(z)\right)^2 dz$$

Examples of estimated γ

Table 1: Shape parameter for the approximated distribution with $\sigma_1 = 1$

(α, β, δ)	γ_{Kurt}	γ_{CDF}	γ_{Tail}			
			$n = 0$	1	2	3
$(0.5, 0.5, 1)$	0.626	0.467	0.768	0.673	0.624	0.642
$(0.5, 0.5, 2)$	0.604	0.492	0.762	0.656	0.603	0.584
$(0.5, 0.7, 2)$	0.633	0.501	0.861	0.741	0.636	0.834
$(0.5, 1.2, 1)$	0.779	0.602	1.160	1.053	0.757	1.165
$(1.5, 1.5, 2)$	1.673	1.373	1.738	1.702	1.683	1.664
$(1.5, 2.5, 1)$	1.908	1.391	1.979	1.959	1.952	1.887
$(1.5, 2.5, 2)$	1.753	1.443	1.842	1.799	1.771	1.741
$(2.5, 3, 3)$	2.295	1.941	2.226	2.261	2.267	2.335

- ✓ γ_{Kurt} and γ_{Tail} are close to each other (Kurtosis measures the heavy tail)
- ✓ γ_{CDF} is little far from other methods $\Rightarrow$ Each method can be used according to the scenario in case.

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Impact of Kurtosis and CDF approaches on the PDF

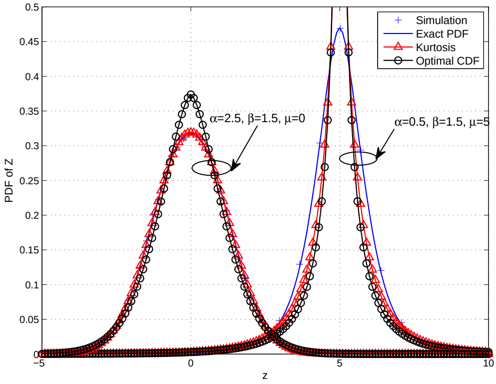

Figure 2: Exact and approximated PDF of the sum of two GGRV. $\beta = 1.5$, $\delta = 2$, and $\sigma_1 = 1$.

- ✓ Perfect match b/w simulated and exact PDF
- ✓ Good tail approximation for both methods
- ✓ $\alpha < 2$: Tail matching for both methods, but big difference around the mean
- ✓ $\alpha > 2$: Better PDF matching using Kurtosis approach than using CDF approach which approximates the CDF rather than the PDF.

Impact of estimation approaches on the CDF

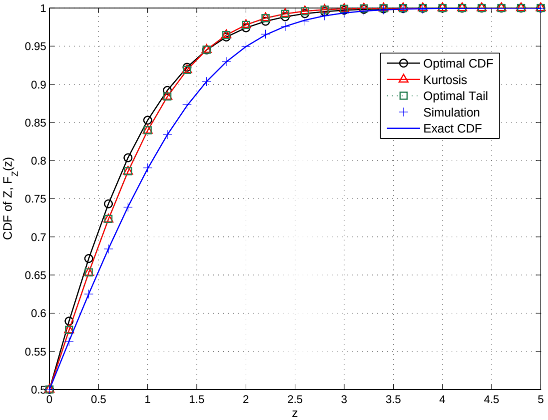

Figure 3: CDF of the sum using Kurtosis, optimal tail, and optimal CDF approaches. $\alpha = 2.5$, $\beta = 1.5$, $\delta = 2$, and $\sigma_1 = 1$

- ✓ Kurtosis and Tail ($n = 3$) methods give very close results to the exact CDF
- ✓ In linear scale, all methods are close to the CDF at saturation, i.e. $F_Z(z) \approx 1$.
- ✓ Plot the gap in Log scale.

Comparison of the Complementary CDF

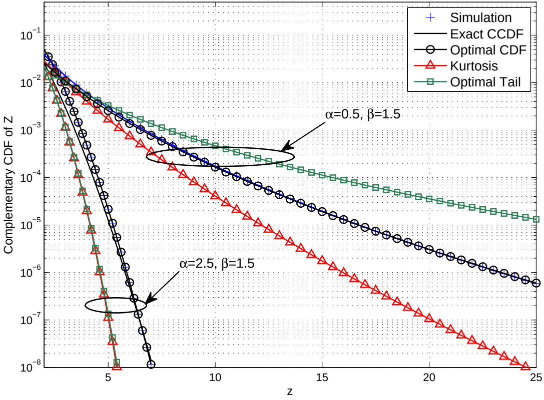

Figure 4: Complementary CDF of the sum of two GGRV. $\beta = 1.5$ and $\delta = 2$.

- ✓ CDF approach matches exact CDF of the sum for both values of α
- ✓ $\alpha < 2$ Kurtosis and Tail methods are far from the exact CDF
- ✓ $\alpha > 2$ The gap decreases and these methods become close to the exact (as seen in Fig.2).

Conclusion

- ✓ Expression of the statistics of the GGD
- ✓ Derivation of the distribution of the sum of two independent GG random variables
- ✓ Expression of the statistics of the sum distribution in terms of the FHF and BFHF
- ✓ Approximation of the distribution of the sum by single GGD using 3 methods
- ✓ Different behaviors of these 3 approaches regarding the exact distribution
- ✓ Choose of the suitable approach depends on the application in case

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Thank you for your attention

Questions ?

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