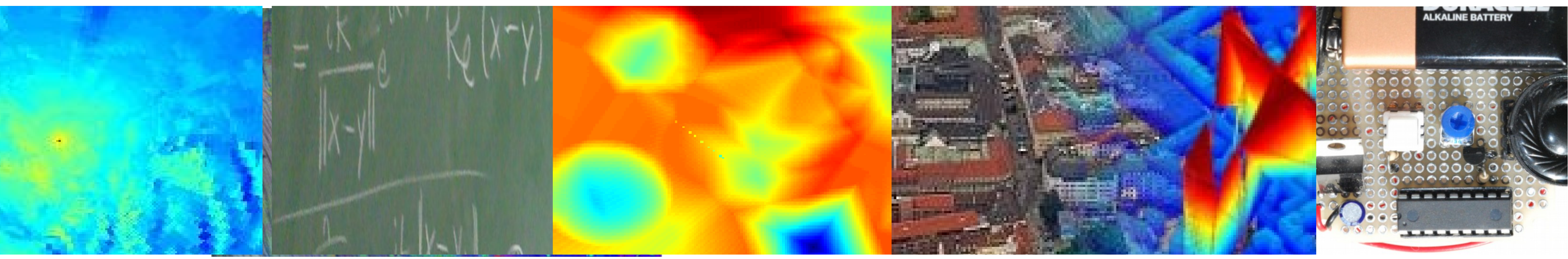
Introduction *to the* Special Session *on* Topological Data Analysis



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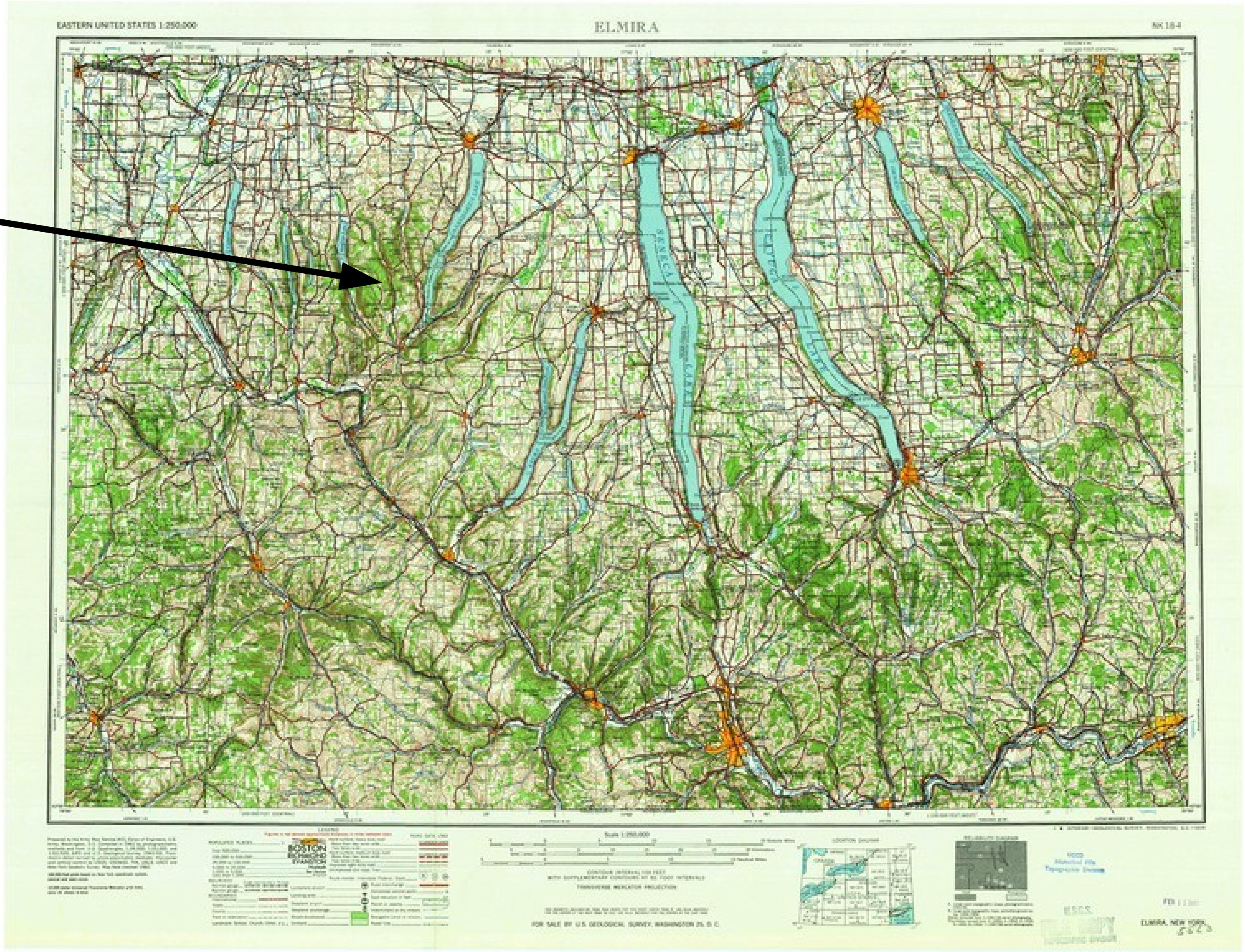
FLORIDA POLYTECHNIC UNIVERSITY

AU

NC STATE UNIVERSITY

What is topology?

Not this!
This is
TopoGRAPHY!



(Thanks USGS!)

Michael Robinson

What is topology?

=

Topology is the study of spaces under continuous deformations

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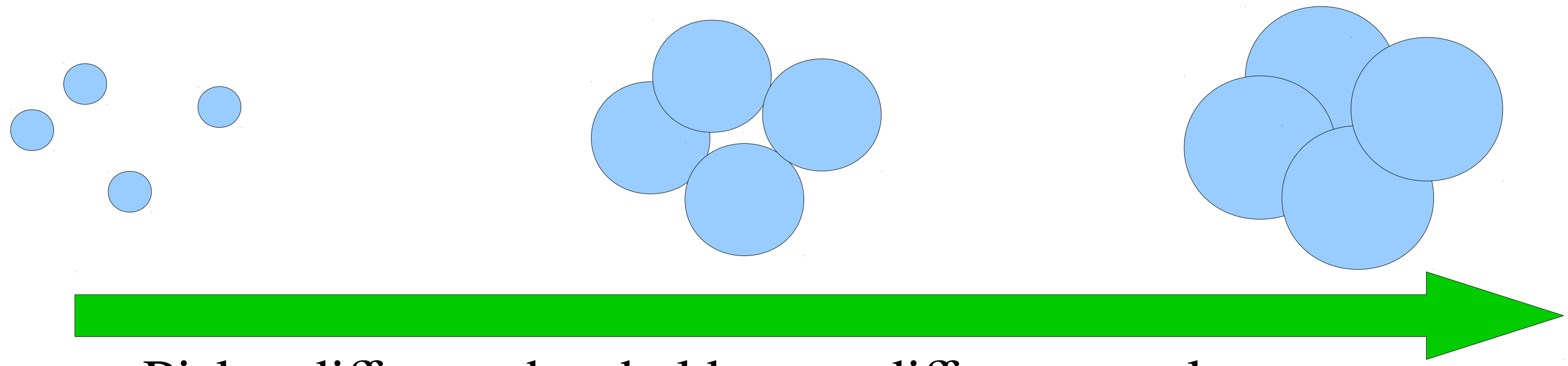
Topology and the curse of thresholds

- Although topology is very flexible, it **also** seems quite brittle. And in signal processing, that's bad!



Pick a different threshold… get different topology

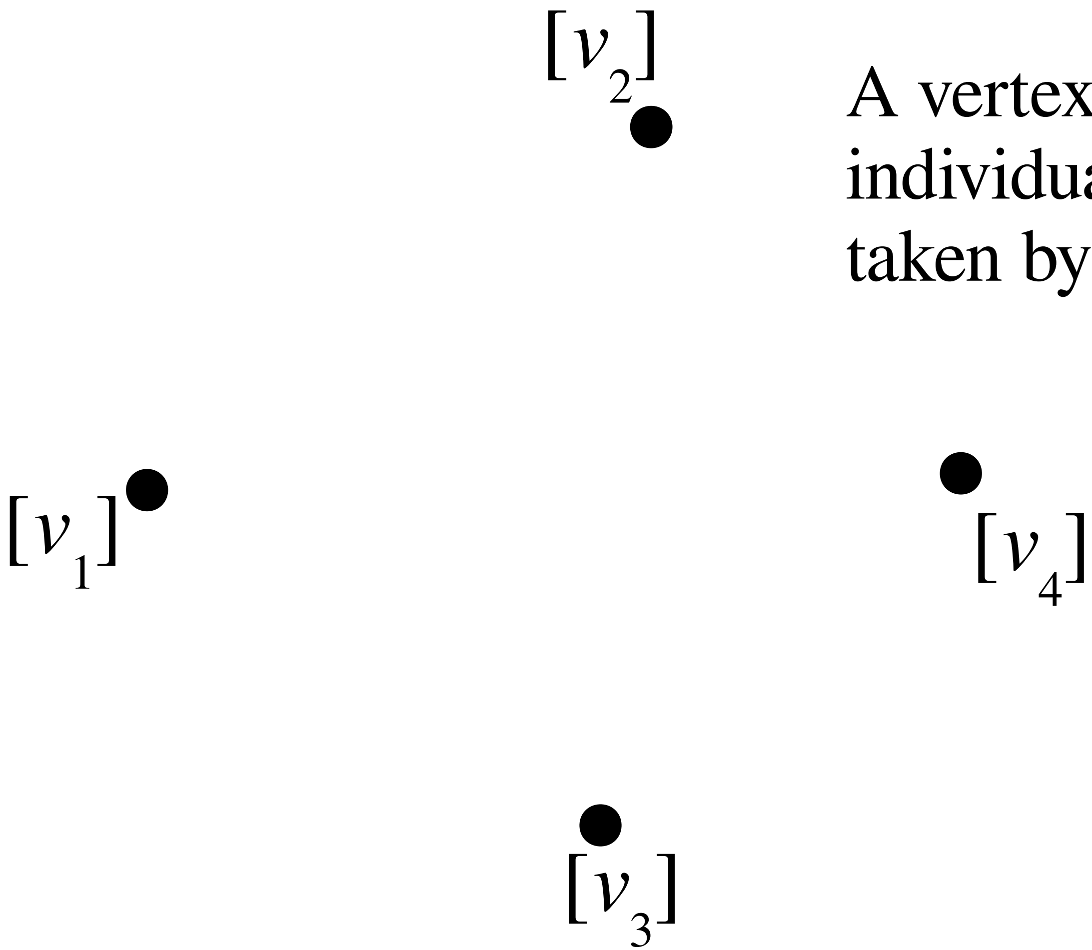
- But a nice idea *persists*… and in the end prevails

Herbert Edelsbrunner, David Letscher, and Afra Zomorodian, Topological persistence and simplification, *Discrete Comput. Geom*. 28 (2002), no. 4, 511–533.

- Rather than selecting one threshold, let's use them **all**!

Simplicial complexes

- A simplicial complex is a collection of *vertices* and ...

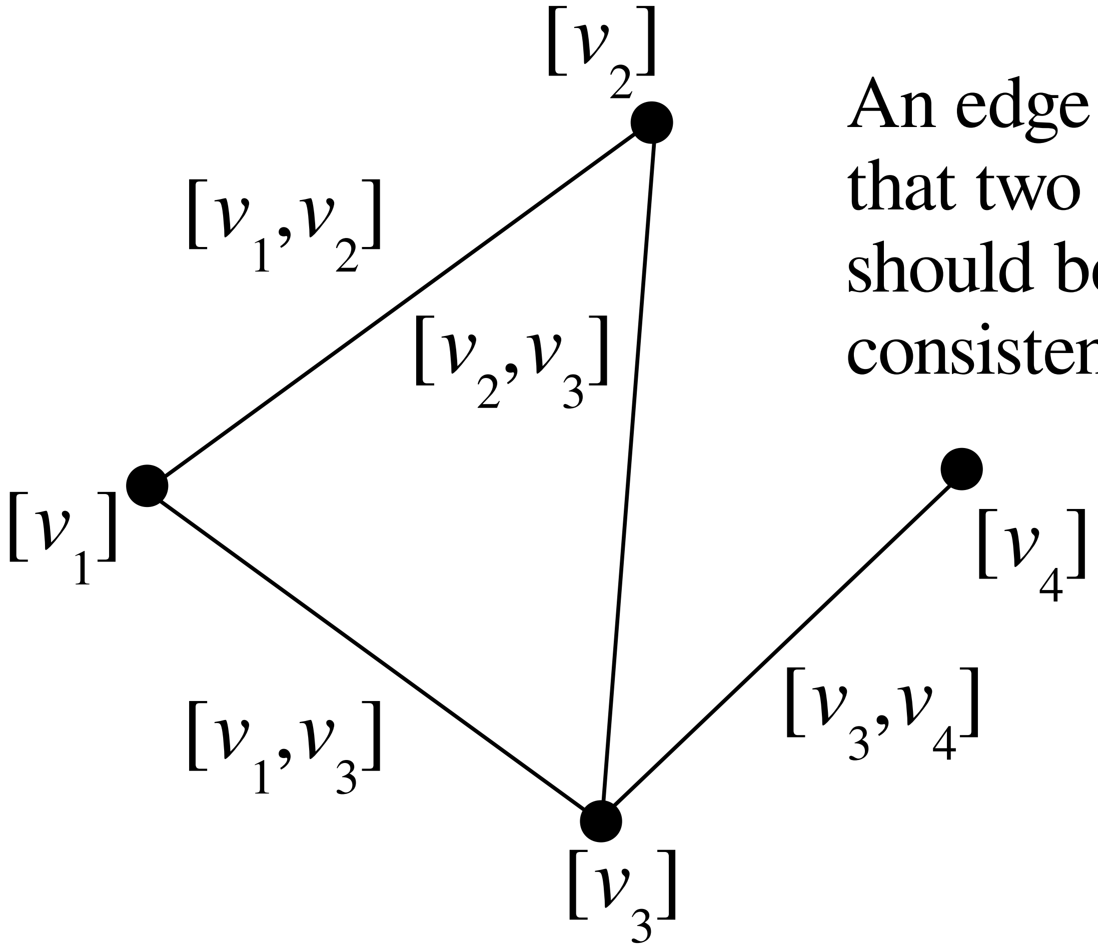


A vertex represents an individual measurement taken by a sensor

Michael Robinson

Simplicial complexes

- … *edges* (pairs of vertices) and ...



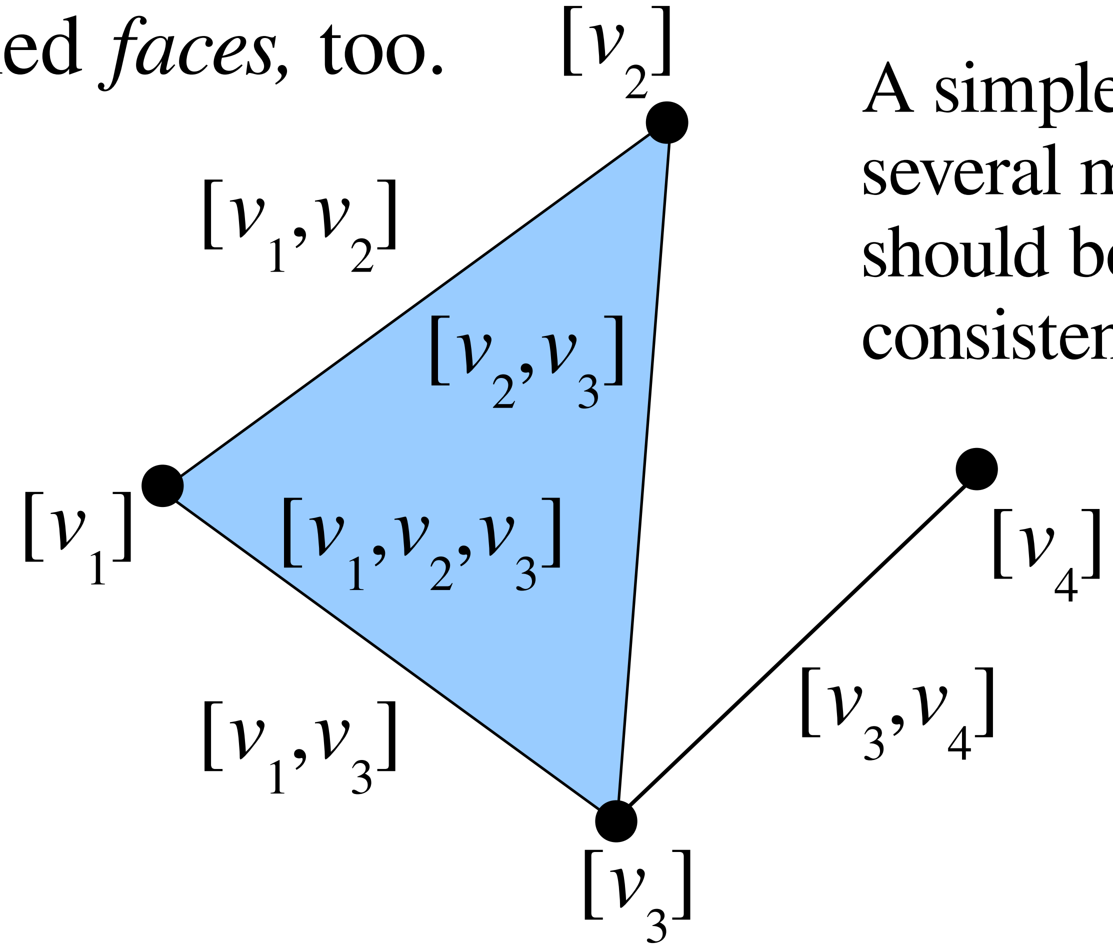
An edge represents the fact that two measurements should be tested for consistency

Usually people call this a *graph*; I will too

Michael Robinson

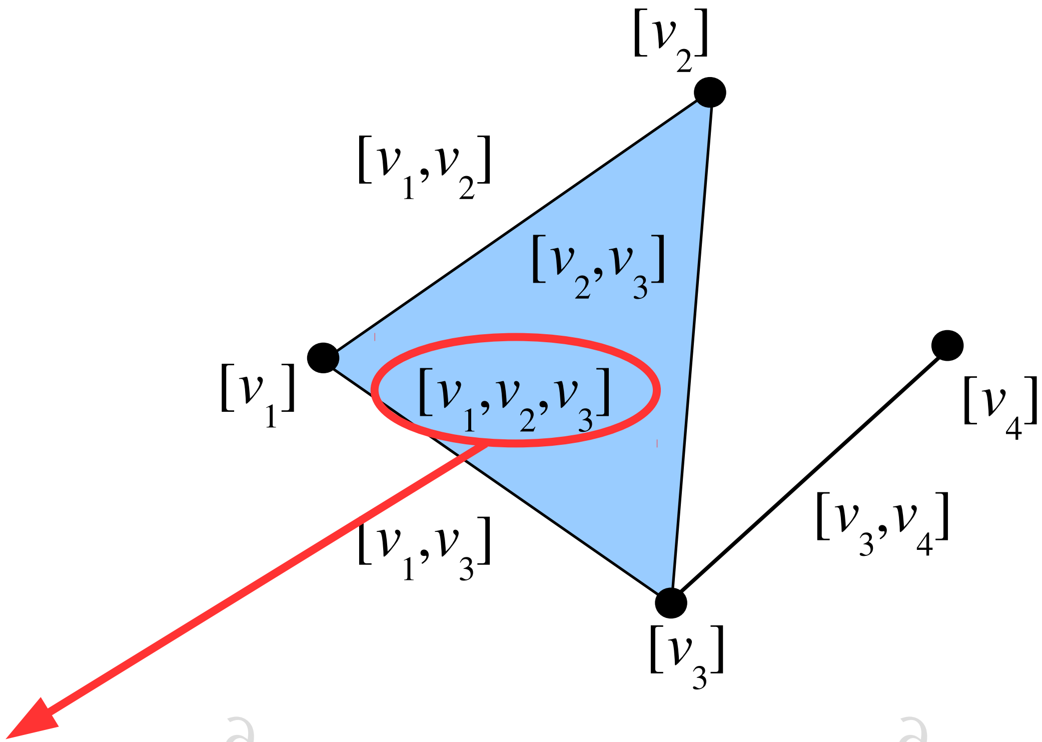
Simplicial complexes

- ... higher dimensional *simplices* (tuples of vertices)
- Whenever you have a simplex, you have all subsets, called *faces*, too.



A simplex represents that several measurements should be tested for consistency

Simplicial chain complex



$$\xrightarrow{\partial_3} \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^4 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{\partial_0} 0$$

	[v_1,v_2,v_3]
[v_1,v_2]	+1
[v_1,v_3]	-1
[v_2,v_3]	+1
[v_3,v_4]	0

	[v_1,v_2]	[v_1,v_3]	[v_2,v_3]	[v_3,v_4]
[v_1]	-1	-1	0	0
[v_2]	+1	0	-1	0
[v_3]	0	+1	+1	-1
[v_4]	0	0	0	+1

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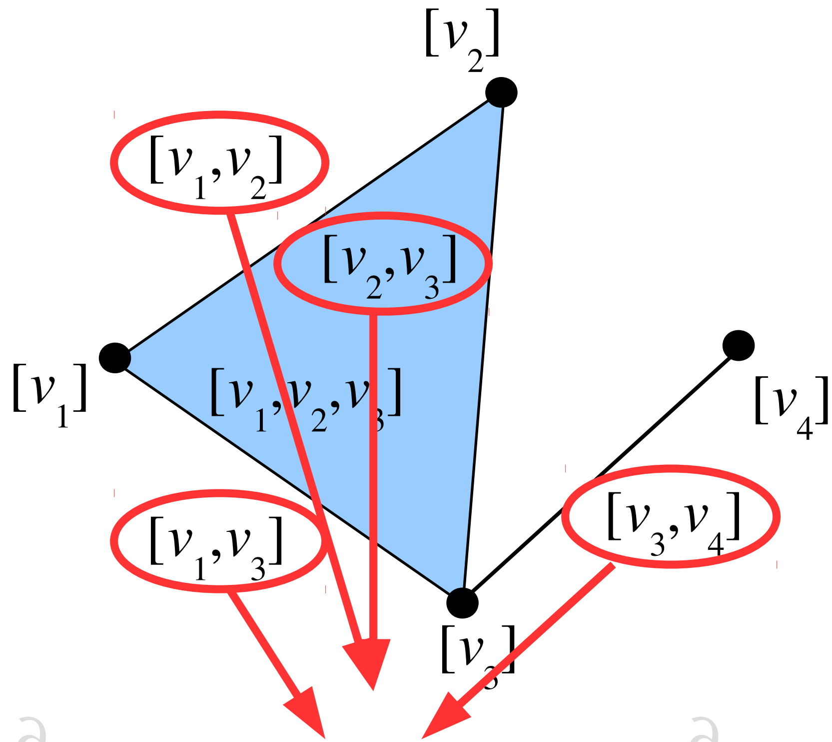
Simplicial chain complex

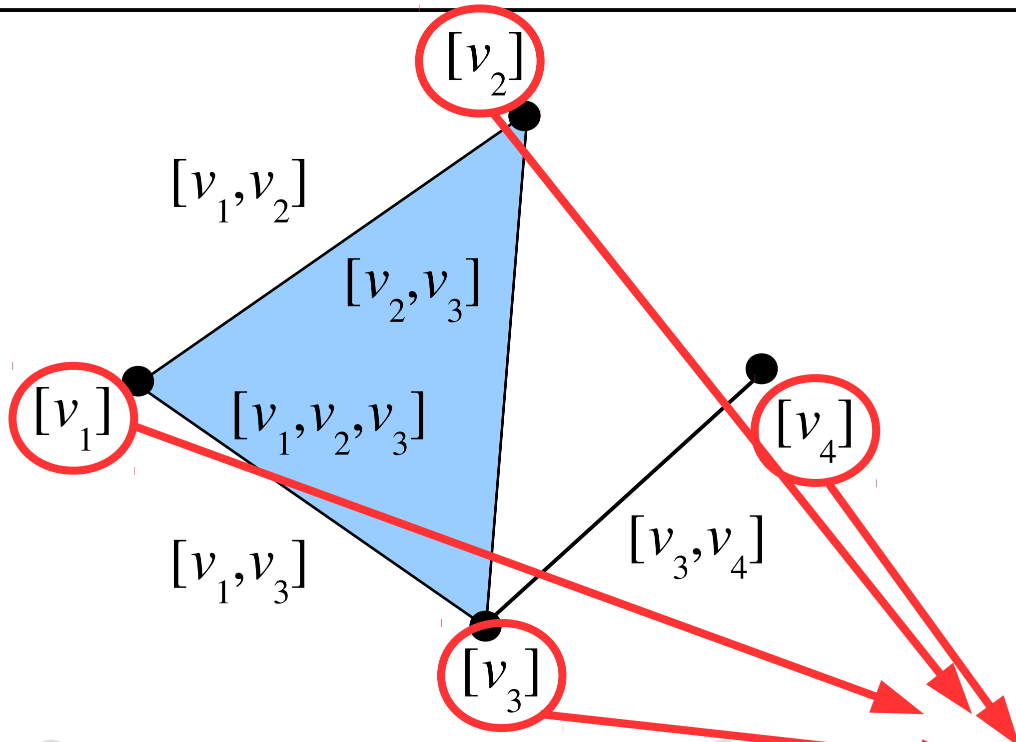


$$\xrightarrow{\partial_3} \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^4 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{\partial_0} 0$$

$$\partial_2: \quad \begin{array}{c} \\ [v_1,v_2] \\ [v_1,v_3] \\ [v_2,v_3] \\ [v_3,v_4] \end{array} \begin{array}{c} [v_1,v_2,v_3] \\ \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \end{pmatrix} \end{array}$$

$$\partial_1: \quad \begin{array}{c} \\ [v_1] \\ [v_2] \\ [v_3] \\ [v_4] \end{array} \begin{array}{c} \begin{array}{cccc} [v_1,v_2] & [v_1,v_3] & [v_2,v_3] & [v_3,v_4] \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix} \end{array}$$

Simplicial chain complex



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	[v_1,v_2]	[v_1,v_3]	[v_2,v_3]	[v_3,v_4]
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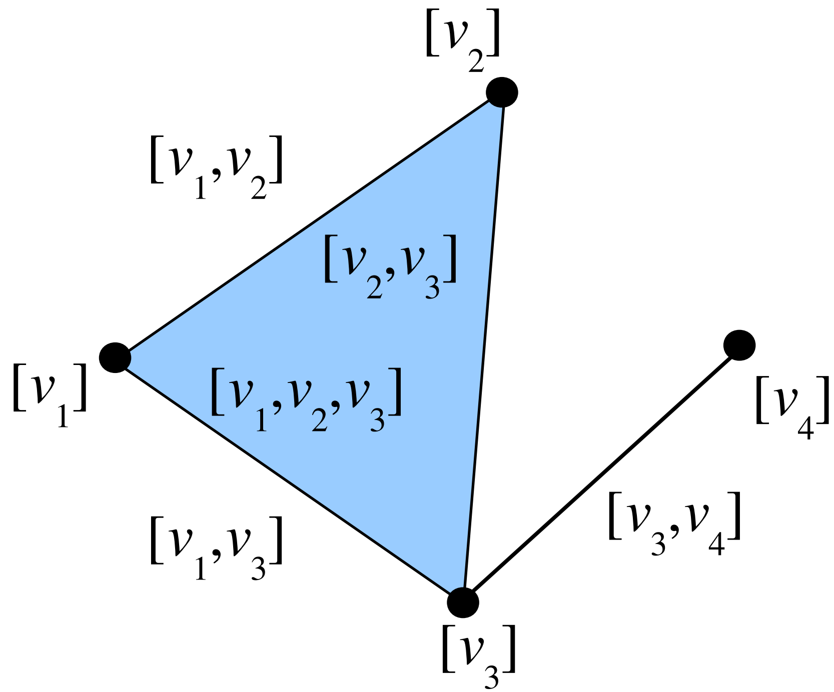
Simplicial chain complex



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∂_2:

	$[v_1,v_2,v_3]$
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∂_1:

	$[v_1,v_2]$	$[v_1,v_3]$	$[v_2,v_3]$	$[v_3,v_4]$
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$[v_4]$	0	0	0	+1

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Simplicial chain complex

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix} \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{\partial_3} \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^4 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{\partial_0} 0$$

∂_2: column $[v_1,v_2,v_3]$; rows $[v_1,v_2]$, $[v_1,v_3]$, $[v_2,v_3]$, $[v_3,v_4]$

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$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

Michael Robinson

Simplicial chain complex

$$\text{image } \partial_2 = \text{image}\begin{pmatrix}+1\\-1\\+1\\0\end{pmatrix} \subseteq \text{kernel } \partial_1 = \text{kernel}\begin{pmatrix}-1 & -1 & 0 & 0\\+1 & 0 & -1 & 0\\0 & +1 & +1 & -1\\0 & 0 & 0 & +1\end{pmatrix}$$

$$\begin{pmatrix}-1 & -1 & 0 & 0\\+1 & 0 & -1 & 0\\0 & +1 & +1 & -1\\0 & 0 & 0 & +1\end{pmatrix}\begin{pmatrix}+1\\-1\\+1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

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Homology of a chain complex

$$\text{image } \partial_2 = \text{image} \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \end{pmatrix} \subseteq \text{kernel } \partial_1 = \text{kernel} \begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$H_2 = \ker \partial_2 / \text{img } \partial_3 \qquad H_1 = \ker \partial_1 / \text{img } \partial_2 \qquad H_0 = \ker \partial_0 / \text{img } \partial_1$$

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Homology of a chain complex

$$\text{image } \partial_2 = \text{image} \begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \end{pmatrix} \subseteq \text{kernel } \partial_1 = \text{kernel} \begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

BUBBLES	LOOPS	COMPONENTS
$H_2 = \ker \partial_2 / \operatorname{img} \partial_3$	$H_1 = \ker \partial_1 / \operatorname{img} \partial_2$	$H_0 = \ker \partial_0 / \operatorname{img} \partial_1$

$$\xrightarrow{\partial_3} \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^4 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{\partial_0} 0$$

∂_2: column $[v_1,v_2,v_3]$; rows $[v_1,v_2]$, $[v_1,v_3]$, $[v_2,v_3]$, $[v_3,v_4]$

$$\begin{pmatrix} +1 \\ -1 \\ +1 \\ 0 \end{pmatrix}$$

∂_1: columns $[v_1,v_2]$, $[v_1,v_3]$, $[v_2,v_3]$, $[v_3,v_4]$; rows $[v_1]$, $[v_2]$, $[v_3]$, $[v_4]$

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

Persistent homology

Goal: obtain a filtration of spaces from a finite pseudometric space X

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X$$

Tactic: Vietoris-Rips complexes

$VR_\varepsilon(X)$ = set of all subsets of X with diameter ε or less

- This is an abstract simplicial complex, and

$$VR_\varepsilon(X) \subseteq VR_\eta(X) \text{ if } \varepsilon \leq \eta$$

Persistent homology

Goal: obtain a filtration of spaces from a finite pseudometric space X

$$H_k(X_0) \rightarrow H_k(X_1) \rightarrow H_k(X_2) \rightarrow \ldots \rightarrow H_k(X)$$

Tactic: Vietoris-Rips complexes

$\mathrm{VR}_\varepsilon(X)$ = set of all subsets of X with diameter ε or less

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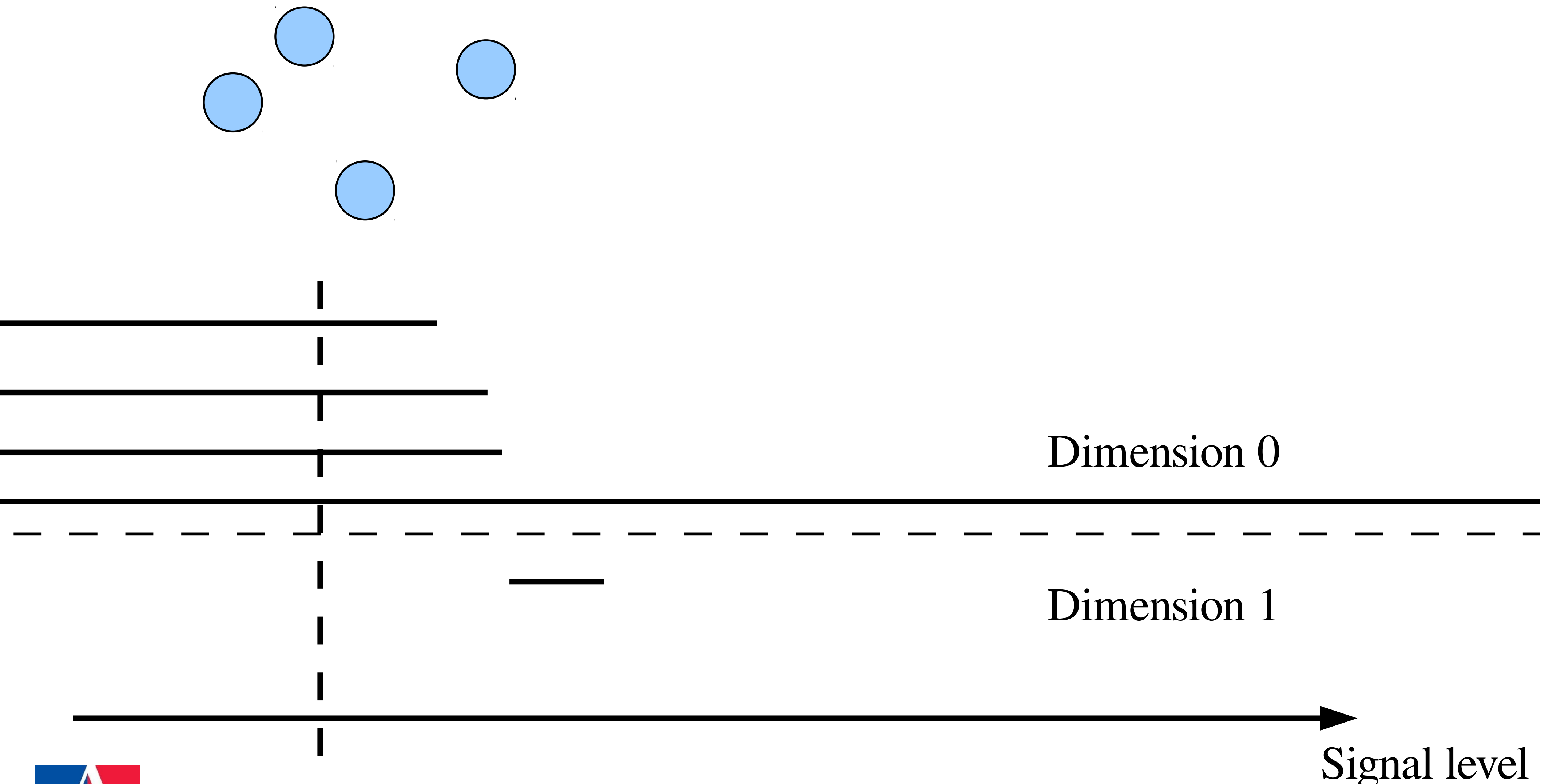
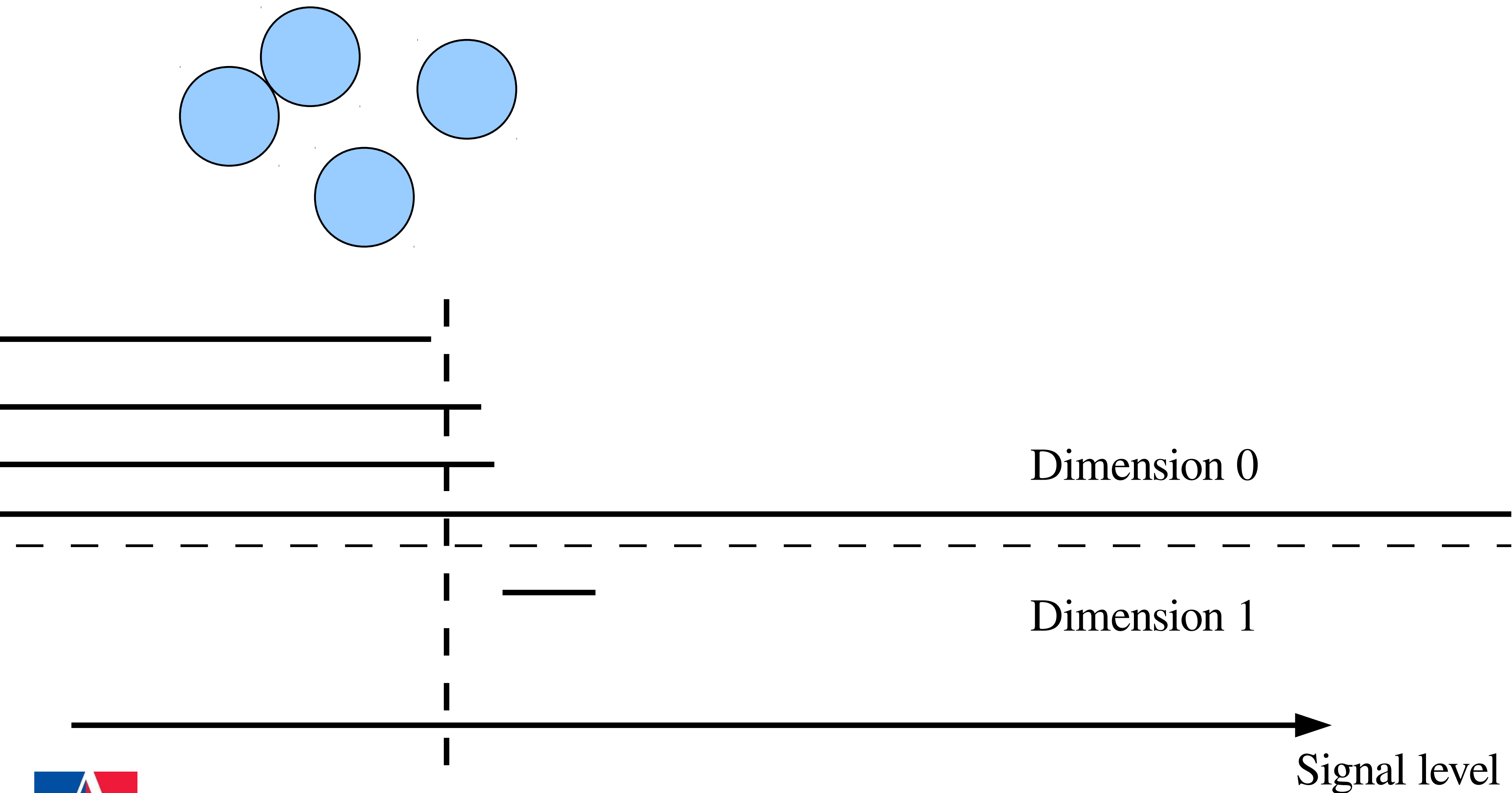
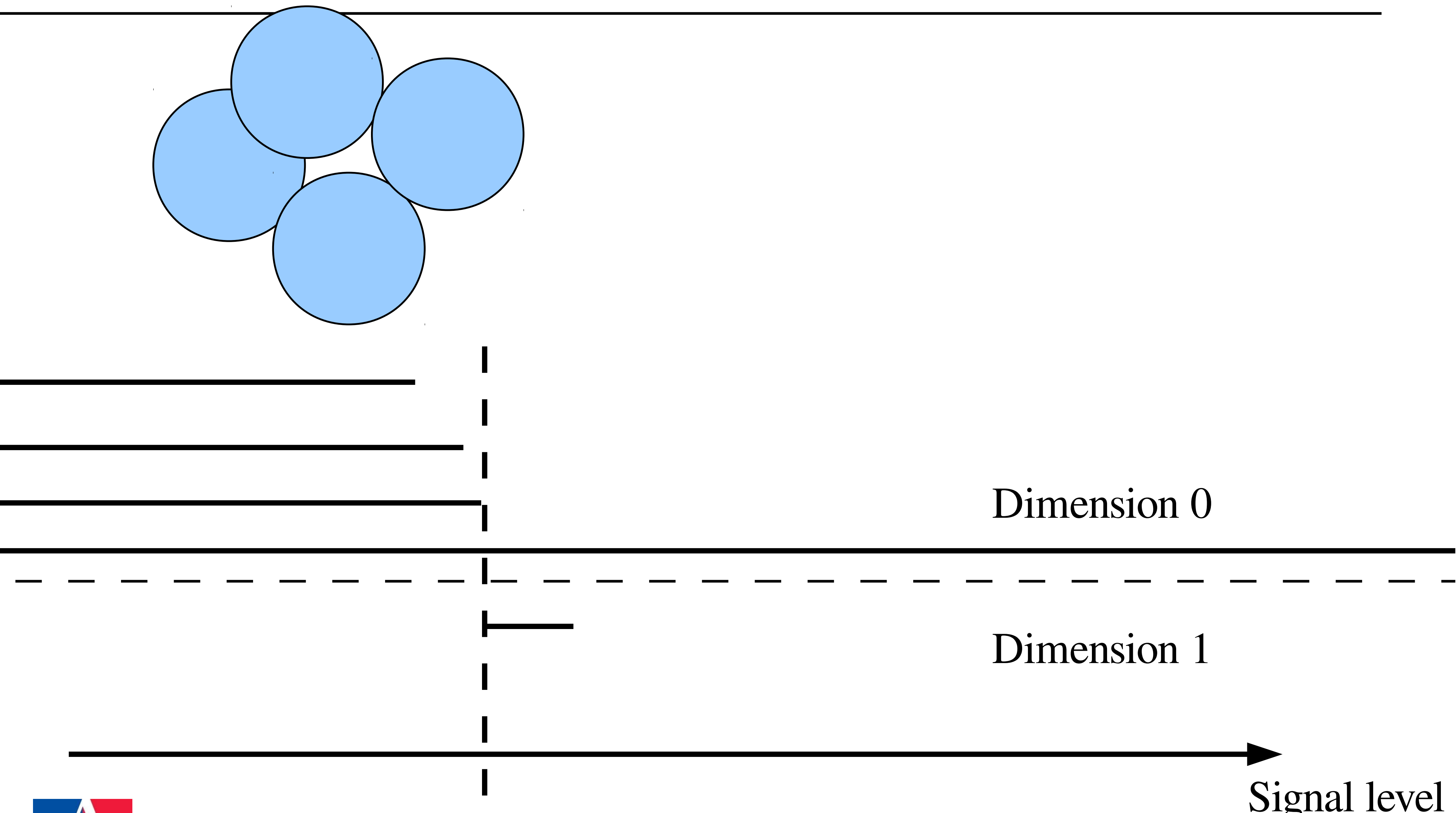
Persistence and model robustness



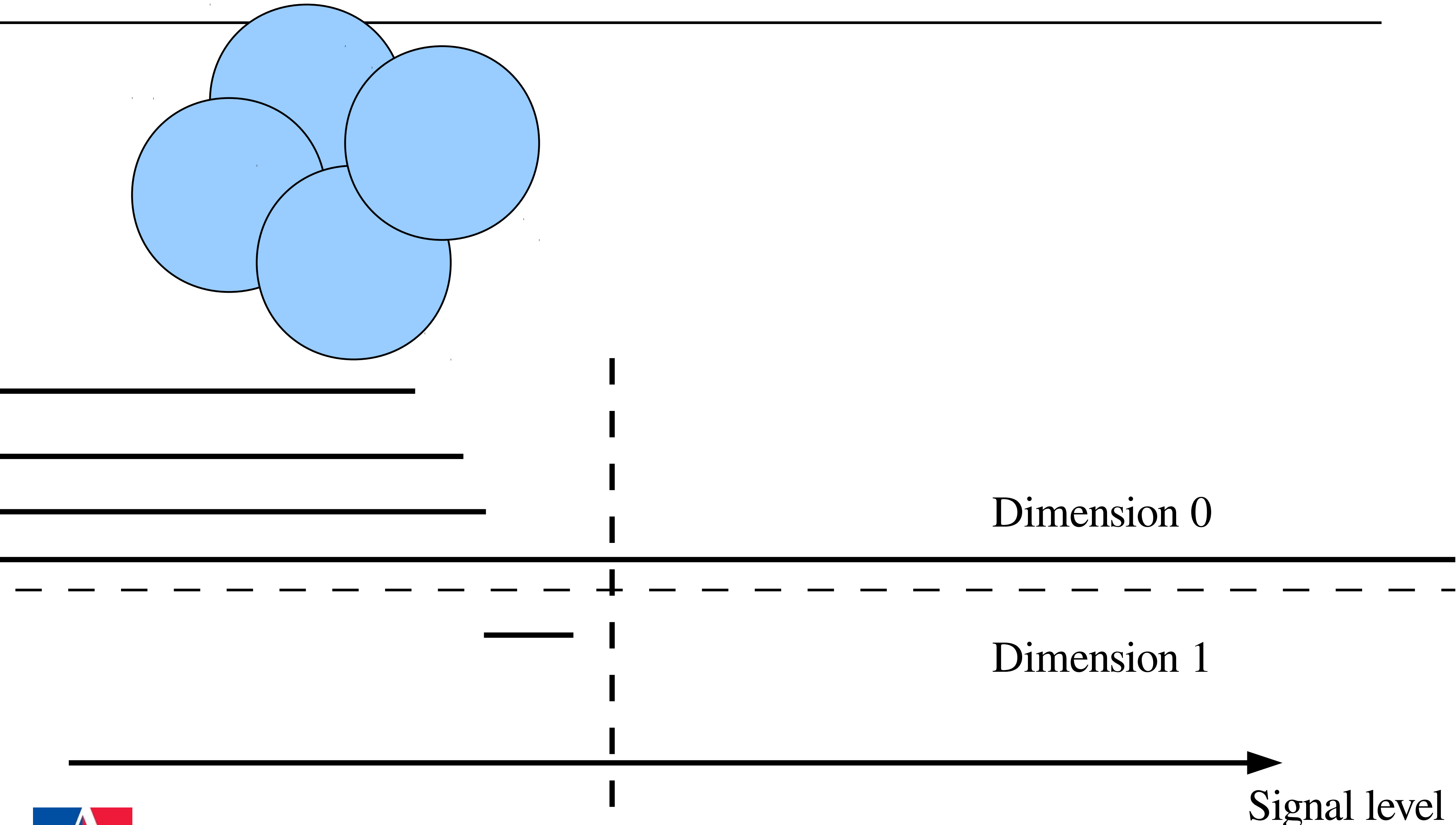
Persistence and model robustness



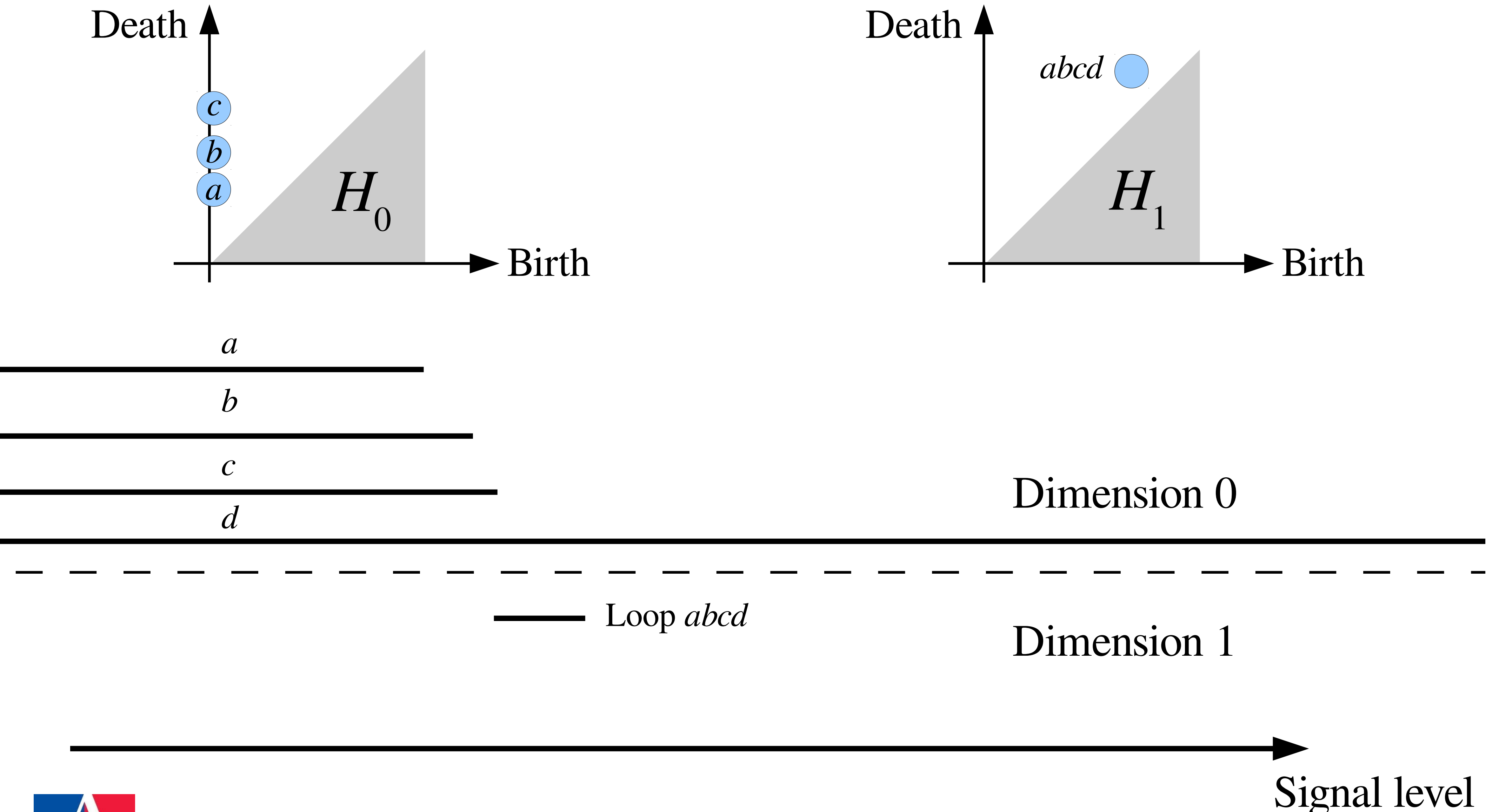
Persistence and model robustness



Persistence and model robustness



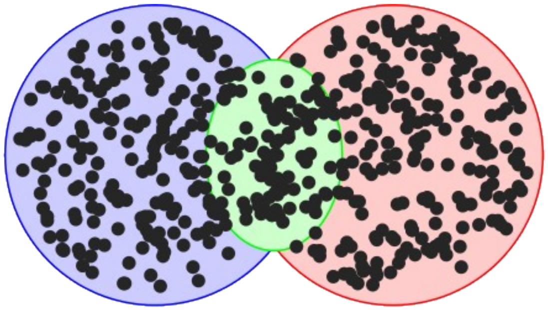
Persistence diagrams



This sessions' talks

Hierarchical Overlapping Clustering

Fernando Gama, Santiago Segarra, Alejandro Ribeiro



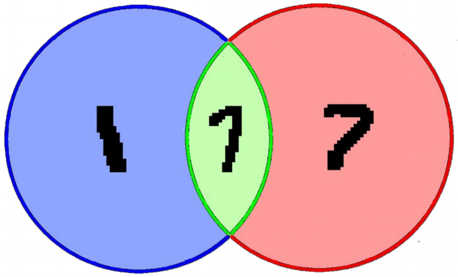
- **Clustering:** Partition dataset. But sets might not admit a partition.
- Where should the green points go?

- **Coverings:** Points can be classified in more than one cluster.

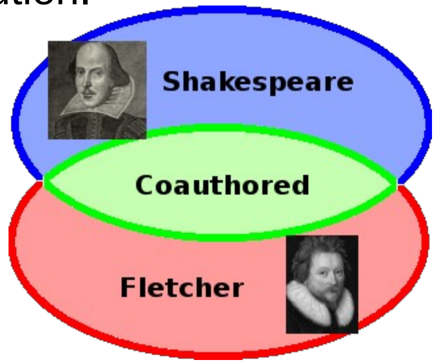
- Hierarchical Overlapping Clustering through Cut Metrics.
- Non-overlapping Clustering: Each point in only one cluster. Partition.
- Hierarchical Clustering: Collection of partitions. Resolution of clusters.
- Ultrametrics: Determines resolution at which nodes are clustered together.
- Non-overlapping Clustering → Hierarchical Clustering → Ultrametrics

- Convex combination of ultrametrics → Cut Metrics
- Cut Metrics: Resolution at which nodes are grouped together. No transitivity.
- Nested coverings: Collection of coverings.
- Covering: Nodes can be in more than one group.
- Cut Metrics → Nested Coverings (Hierarchical) → Covering

- **Results:** MNIST Handwritten Digits. Authorship Attribution.



Identify ambiguous digits
When classifying 1 and 7



Classify plays by author
and co-authored plays

Distances between Directed Networks and Applications

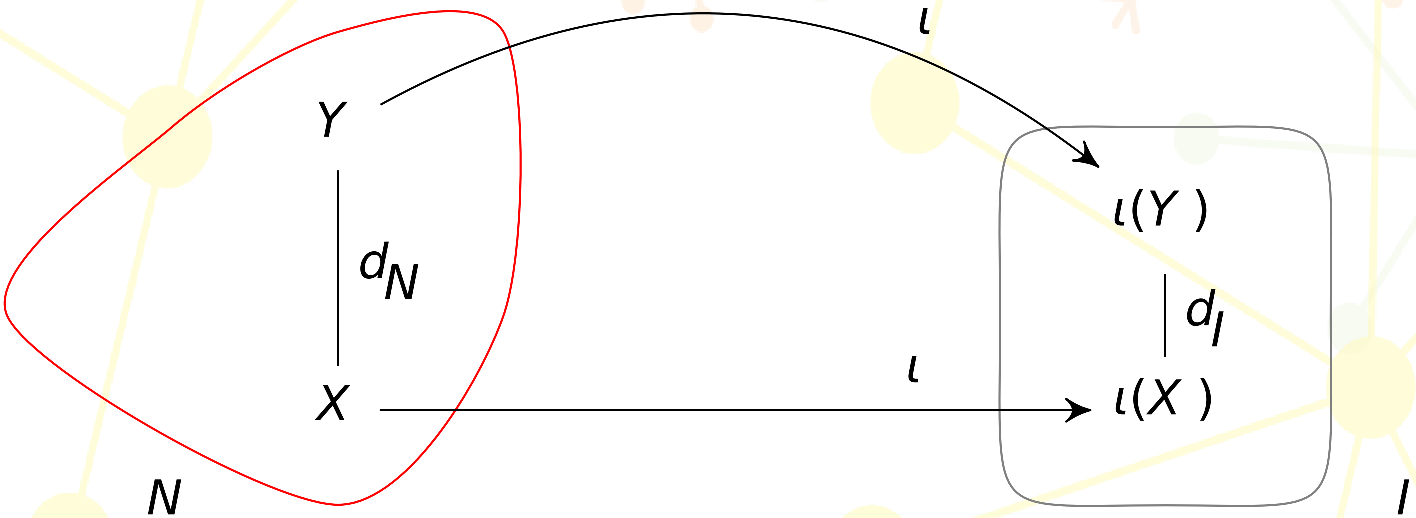
Facundo Mémoli, joint work with Samir Chowdhury

Background:

- Data sets containing asymmetric edge relations can be interpreted as directed, weighted networks.
- A central goal of network analysis is to develop metrics that efficiently compute dissimilarity between networks.

Our Contributions:

- A legitimate metric between directed, weighted networks.
- Easily computable invariants to test for dissimilarity between networks.
- Lower bounds for the network distance, based on these invariants.



Hypergraph signal processing

Sergio Barbarossa and Mikhail Tsitsvero

Idea: Hypergraphs generalize graphs and simplicial complexes

All cliques are simplices → Flag complex → Abstract simplicial complex

Vertices and edges only → Undirected graph → Abstract simplicial complex

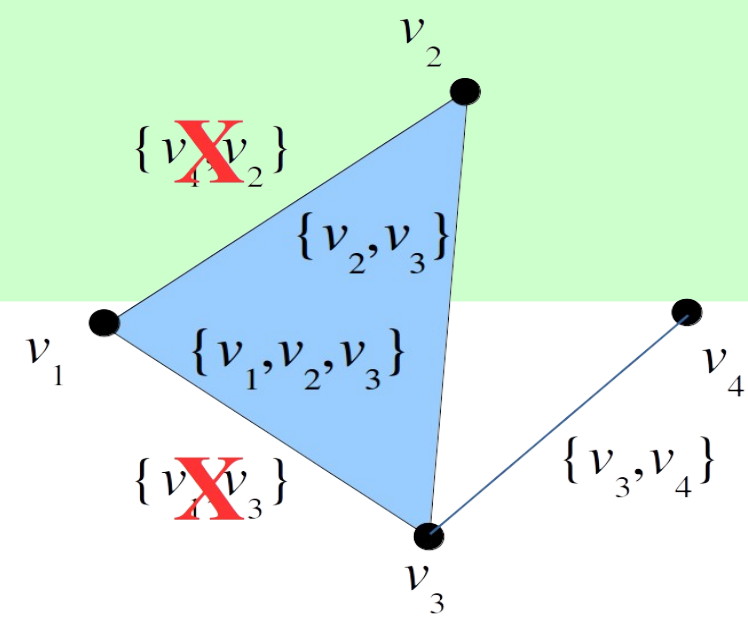
Simplices are defined by their vertices, simplices always have all their faces

Abstract simplicial complex

↓

Undirected hypergraph

Hyperedges defined by vertices, but other than that, anything goes...

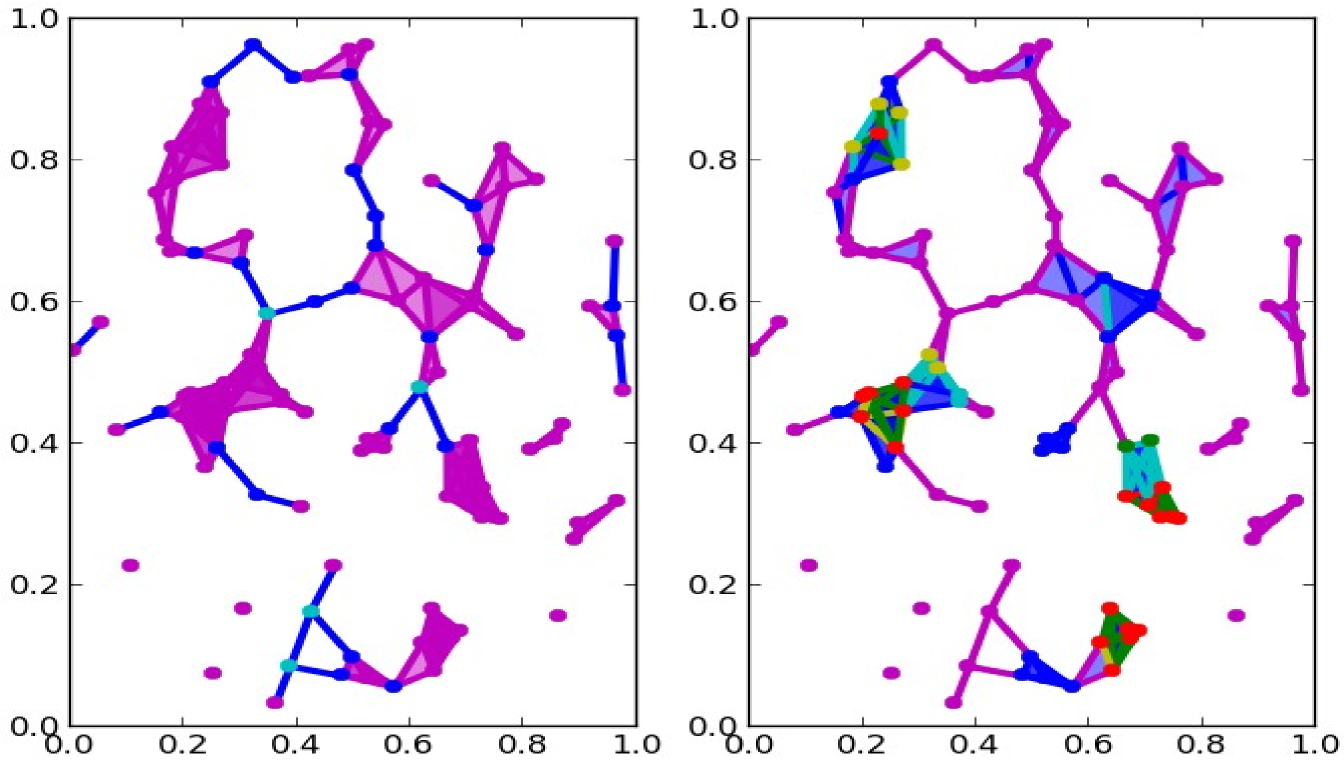


Persistent local homology

Brittany Fasy and Bei Wang

Idea: local homology detects stratifications!

Magenta = 0
Blue = 1
Cyan = 2
Green = 3
Yellow = 4
Red = 5+

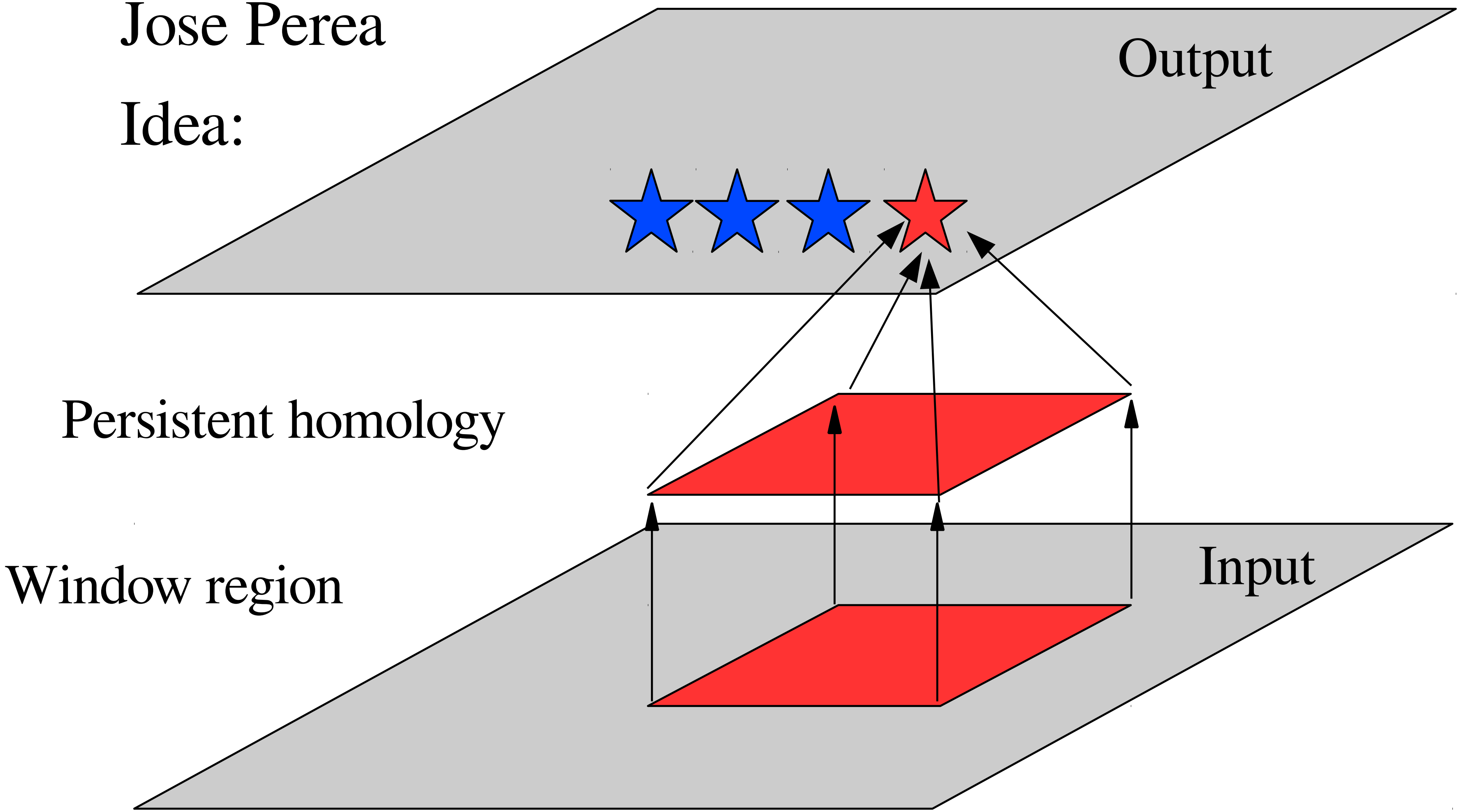


Local H_1 Local H_2

Persistent homology and sliding windows

Jose Perea

Idea:



For more information

...or if you need anything...

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