Extension of Nested Arrays with the Fourth-Order Difference Co-Array Enhancement

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1. Introduction

- Direction-of-arrival (DOA) estimation is an important area in array signal processing.

- Co-array equivalence plays an important role in designing sparse array structures, leading to an effective solution for underdetermined DOA estimation.

- Among all these sparse array structures based on the difference co-array concept, nested arrays and co-prime arrays are two classes of most notable sparse structures with systematic ways for convenient structure construction.

- Both the spatial smoothing based subspace methods and compressive sensing (CS) based sparse signal reconstruction methods can be employed for underdetermined DOA estimation.

1. Introduction

- Previous research:
  - Most of the work about DOA estimation for the nested arrays and co-prime arrays are based on the second-order difference co-array concept.
  - The virtual array concept for the fourth-order cumulants and higher order cumulants based DOA estimation have been proposed. Based on the $2q$-th order cumulants, the $2q$-th order difference co-array concept is proposed.
  - To exploit the $2q$-th order difference co-array concept, multiple level nested arrays are proposed with a substantial increase in the number of degrees of freedom (DOFs), and spatial smoothing based subspace method is applied to find the DOAs.

1. Introduction

- Our contributions:
  - By reviewing the two-level nested arrays, the fourth-order difference co-arrays are further optimized by exploring the physical array aperture and the symmetric features in the second-order difference co-array, which have not been fully exploited in the construction of the existing four-level nested arrays.
  - The number of DOFs of the proposed extended construction is always larger than the standard two-level nested array, and it is also larger than that of the four-level nested array for less than 21 sensors.
  - For 20 physical sensors with our proposed structure, the number of virtual ULA sensors at the fourth-order difference co-array stage can be 2223, which is sufficient for most applications.

2. DOA Estimation for Four-Level Nested Arrays

- The set of sensor positions of a general N-sensor linear array is expressed as $S = \left\{p_0 \cdot d, p_1 \cdot d, \ldots, p_{N-1} \cdot d\right\}$.

- Assume that there are K mutually uncorrelated far-field narrowband signals $s_k(t)$ impinging from incident angles θ_k, $k = 1, 2, \ldots, K$, respectively. After sampling, the array output model is obtained.

$$\mathbf{x}[i] = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}[i] + \overline{\mathbf{n}}[i] \ . \tag{1}$$

- The steering matrix $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$, with its k-th column vector $\mathbf{a}(\theta_k)$ being the steering vector corresponding to the k-th source signal, expressed as

$$\mathbf{a}(\theta_k) = \left[e^{-j\frac{2\pi p_0 d}{\lambda}\sin(\theta_k)}, \ldots, e^{-j\frac{2\pi p_{N-1} d}{\lambda}\sin(\theta_k)}\right]^T \ . \tag{2}$$

2. DOA Estimation for Four-Level Nested Arrays

- The Four-Level Nested Array (FL-NA) has four sub-arrays.
- Denote d as the unit spacing and $N_0 = 0$. For $1 \leq m \leq 3$, the m-th sub-array has N_m sensors located at

$$\left\{ nd \left[\prod_{\tilde{m}=0}^{m-1} (N_{\tilde{m}} + 1) \right], n = 1, 2, \ldots, N_m \right\} , \tag{3}$$

while the sensors of the fourth sub-array with $N_4 + 1$ sensors are located at

$$\left\{ nd \left[\prod_{\tilde{m}=0}^{3} (N_{\tilde{m}} + 1) \right], n = 1, 2, \ldots, N_4 + 1 \right\} . \tag{4}$$

- There are $N = \sum_{m=1}^{4} N_m + 1$ physical sensors in total.

2. DOA Estimation for Four-Level Nested Arrays

- Under the assumption of Gaussian white noise, the fourth-order cumulant matrix of the observed column vector $\mathbf{x}[i]$ for the arrangement indexed by l can be obtained by

$$\mathbf{C}_{4,\mathbf{x}}(l) = \sum_{k=1}^{K} c_{4,s_k} \left[\mathbf{a}(\theta_k)^{\otimes l} \otimes \mathbf{a}(\theta_k)^{*\otimes(2-l)}\right] \times \left[\mathbf{a}(\theta_k)^{\otimes l} \otimes \mathbf{a}(l,\theta_k)^{*\otimes(2-l)}\right]^H , \tag{5}$$

  where $l = 0, 1$. $\mathbf{a}(\theta_k)^{\otimes l}$ denotes $\mathbf{a}(\theta_k) \otimes \ldots \otimes \mathbf{a}(\theta_k)$ with $\mathbf{a}(\theta_k)$ for l times.

- The fourth-order auto-cumulant of source signal $s_k[i]$ can be expressed as

$$c_{4,s_k} = \mathrm{Cum}\left\{s_k[i], s_k[i], s_k^*[i], s_k^*[i]\right\} . \tag{6}$$

2. DOA Estimation for Four-Level Nested Arrays

- We set $l = 1$, and by vectorizing $\mathbf{C}_{4,\mathbf{x}}(1)$ we obtain

$$\mathbf{z} = \mathrm{vec}\left\{\mathbf{C}_{4,\mathbf{x}}(1)\right\} = \mathbf{B}\mathbf{u} \ . \tag{7}$$

- Equation (7) characterises a virtual array, whose equivalent steering matrix $\mathbf{B} = [\mathbf{b}(\theta_1), \ldots, \mathbf{b}(\theta_K)]$ with each column vector $\mathbf{b}(\theta_k) = [\mathbf{a}(\theta_k) \otimes \mathbf{a}(\theta_k)^*]^* \otimes [\mathbf{a}(\theta_k) \otimes \mathbf{a}(\theta_k)^*]$. $\mathbf{u} = \left[c_{4,s_1}, c_{4,s_2}, \ldots, c_{4,s_K}\right]$ is the equivalent signal vector.

- To obtain the DOA results, subspace methods can be applied directly to $\mathbf{C}_{4,\mathbf{x}}(l)$ in (5), and spatial smoothing based subspace methods can be employed in the virtual model characterised by (7).

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- For the given general linear array, the *second-order difference co-array* (also known as *difference co-array*) set is defined as

$$\mathbb{C}_{\mathbb{A}} = \Phi_A \cdot d \ , \tag{8}$$

with the set of difference co-array lags Φ_A given by $\Phi_A = \{p_{n_1} - p_{n_2}\}$, $0 \leq n_1, n_2 \leq N - 1$.

- The set of *fourth-order difference co-array* is defined as

$$\mathbb{C}_{\mathbb{B}} = \Phi_B \cdot d \ , \tag{9}$$

with the set of the fourth-order difference co-array lags $\Phi_B = \{p_{n_1} + p_{n_2} - p_{n_3} - p_{n_4}\}$. Note that $0 \leq n_1, n_2, n_3, n_4 \leq N - 1$.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- The set Φ_B can be rewritten as

$$\Phi_B = \left\{(p_{n_1} - p_{n_3}) - (p_{n_4} - p_{n_2})\right\}. \tag{10}$$

  Note that $(p_{n_1} - p_{n_3}) \in \Phi_A$ and $(p_{n_4} - p_{n_2}) \in \Phi_A$.

- The sets Φ_A and Φ_B of a general Two-Level Nested Array (TL-NA) with $N_1 + N_2$ physical sensors can be expressed as

$$\begin{aligned} \Phi_A &= \left\{\mu, -N_2(N_1+1)+1 \leq \mu \leq N_2(N_1+1)-1\right\}, \\ \Phi_B &= \left\{\mu, -2N_2(N_1+1)+2 \leq \mu \leq 2N_2(N_1+1)-2\right\}. \end{aligned} \tag{11}$$

- The number of consecutive integers is increased to $4N_2(N_1+1)-3$ in Φ_B, which suggest that more DOFs can be exploited by employing the fourth-order difference co-array based method.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- However, the set Φ_A of the TL-NA indicates that the virtual array generated at the difference co-array stage is only a ULA, and the increase in the number of consecutive integers from Φ_A to Φ_B is limited.

- Then, a novel sparse array extension based on the TL-NA is constructed, optimising the consecutive integers at the fourth-order difference co-array stage with each introduced physical sensor of the third sub-array.

- Define the sensor positions of the introduced third sub-array as $\alpha_{n_3}d$, $0 \leq n_3 \leq N_3 - 1$, where N_3 is the sensor number of the third sub-array under construction.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- To maximise the number of consecutive co-array lags, the covered ranges associated with different introduced sensors $\alpha_{n_3} d$ should be adjacent to each other. Then we can obtain

$$\begin{aligned} \alpha_0 &= 4N_2(N_1+1) - 2 \ , \\ \alpha_{n_3} - \alpha_{n_3-1} &= 3N_2(N_1+1) - 2 \ . \end{aligned} \quad (12)$$

- Therefore, the third sub-array is also a uniform linear sub-array with the starting position of $\left[4N_2(N_1+1) - 2\right]d$ and the inter-element spacing $\left[3N_2(N_1+1) - 2\right]d$.

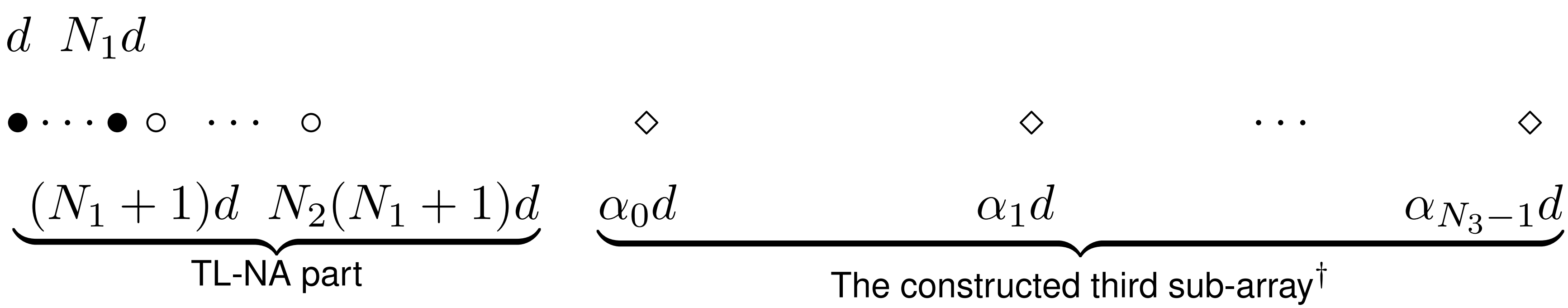
- This larger inter-element spacing is due to exploration of the physical aperture and the symmetric information at the second-order difference co-array stage, which is not exploited in the design of the FL-NA.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- The maximum integer of the fourth-order difference co-array lag $M_0 = (3N_3 + 2)N_2(N_1 + 1) - 2N_3 - 2$. The set of the fourth-order difference co-array lags Φ_B of our proposed structure is updated to

$$\Phi_B = \{\mu, -M_0 \leq \mu \leq M_0\} \ . \tag{13}$$

- A general structure of the proposed array extension:



†$\alpha_0 = 4N_2(N_1 + 1) - 2$, $\alpha_1 = 7N_2(N_1 + 1) - 4$, and $\alpha_{N_3-1} = (3N_3 + 1)N_2(N_1 + 1) - 2N_3$.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- For a FL-NA with $N = \sum_{m=1}^{4} N_m + 1$ physical sensors, the number of consecutive lags at the fourth-order difference co-array stage is

$$2\prod_{m=1}^{4}(N_m + 1) - 1 \ . \tag{14}$$

- In our proposed structure SAFOE-NA, the number of consecutive lags at the fourth-order difference co-array stage is $2M_0 + 1$, where

$$M_0 = 3(N_3 + \tfrac{2}{3})N_2(N_1 + 1) - 2N_3 - 2 \ . \tag{15}$$

  The second term $2N_3$ is much smaller than the first term.

- Then we compare the maximum consecutive lags after applying Arithmetic Mean-Geometric Mean (AM-GM) inequality to find the range in which more DOFs can be provided by our proposed structure.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

- Note N is a positive integer, and the solution can be obtained as

$$1.3739 \leq N \leq 20.6100 \ . \tag{16}$$

- Therefore, for $N \leq 20$, our proposed structure can provide more DOFs than the FL-NA.

- Compared to the TL-NA, our extended structure always gives a significantly larger number of DOFs.

- Furthermore, for 20 physical sensors with $(N_1, N_2, N_3) = (6, 7, 7)$ for our proposed structure, the number of virtual ULA sensors at the fourth-order difference co-array stage is 2223, which is sufficient for most applications.

3. Sparse Array Extension Based on the Fourth-Order Difference Co-Array Concept

Table 1: Comparison of the Fourth-Order Difference Co-Array Lags

Structures	Number of Sensors	Number of Consecutive Lags
TL-NA	$N_1 + N_2$	$4N_2(N_1+1) - 3$
FL-NA	$N = \sum_{m=1}^{4} N_m + 1$	$2\prod_{m=1}^{4}(N_m+1) - 1$
SAFOE-NA	$N = \sum_{m=1}^{3} N_m$	$2M_0 + 1^{\dagger}$

Array Structures	(N_1, N_2), (N_1, N_2, N_3, N_4) or (N_1, N_2, N_3)	Number of Sensors	Number of Consecutive Lags
TL-NA	(2,3)	5	33
FL-NA	(1,1,1,1)	5	31
SAFOE-NA	(1,2,2)	5	53
TL-NA	(8,9)	17	321
FL-NA	(4,4,4,4)	17	1249
SAFOE-NA	(5,6,6)	17	1413
TL-NA	(10,11)	21	481
FL-NA	(5,5,5,5)	21	2591
SAFOE-NA	(7,7,7)	21	2545

† $M_0 = (3N_3 + 2)N_2(N_1 + 1) - 2N_3 - 2.$

4. Simulation Results

- Apart from the spatial smoothing based subspace method, CS-based method can be employed for underdetermined DOA estimation.

- Consider the following examples with a small number of sensors:

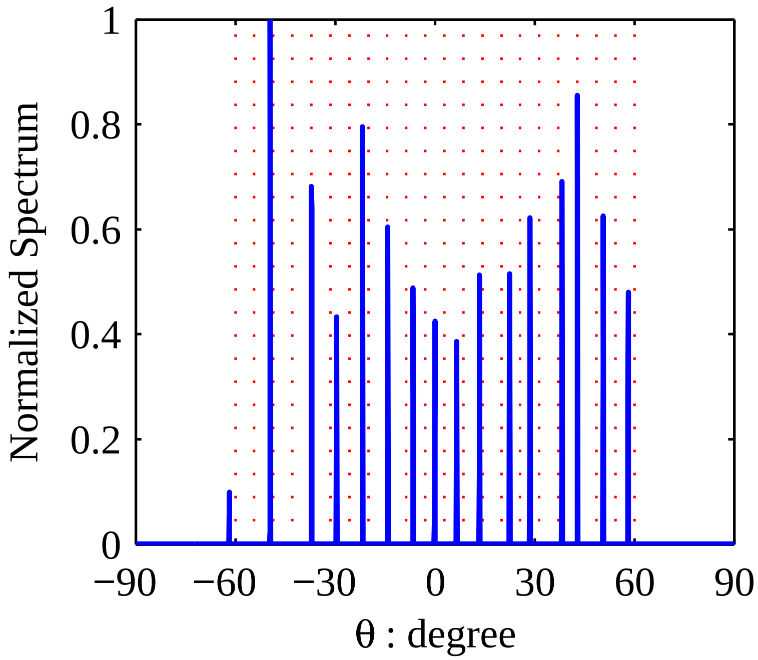
  – The number of physical sensors is set to 5.

  – $(N_1, N_2) = (2, 3)$ for the standard TL-NA, $(N_1, N_2, N_3, N_4) = (1, 1, 1, 1)$ for the FL-NA with $\sum_{m=1}^{4} N_m + 1 = 5$ sensors, and $(N_1, N_2, N_3) = (1, 2, 2)$ for our proposed extended structure SAFOE-NA.

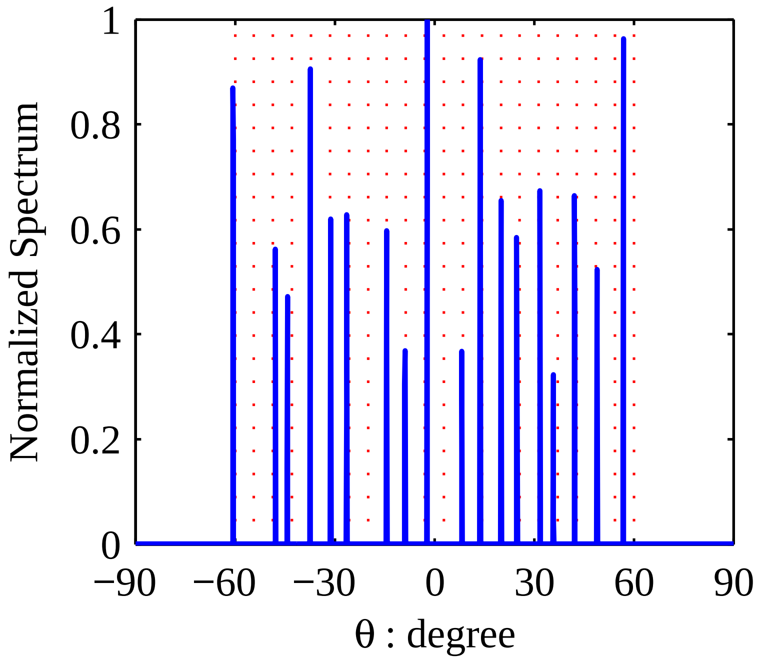
  – The unit spacing $d = \lambda/2$, where λ is the signal wavelength.

4. Simulation Results

- DOA estimation results for different structures (source number $K = 22$):



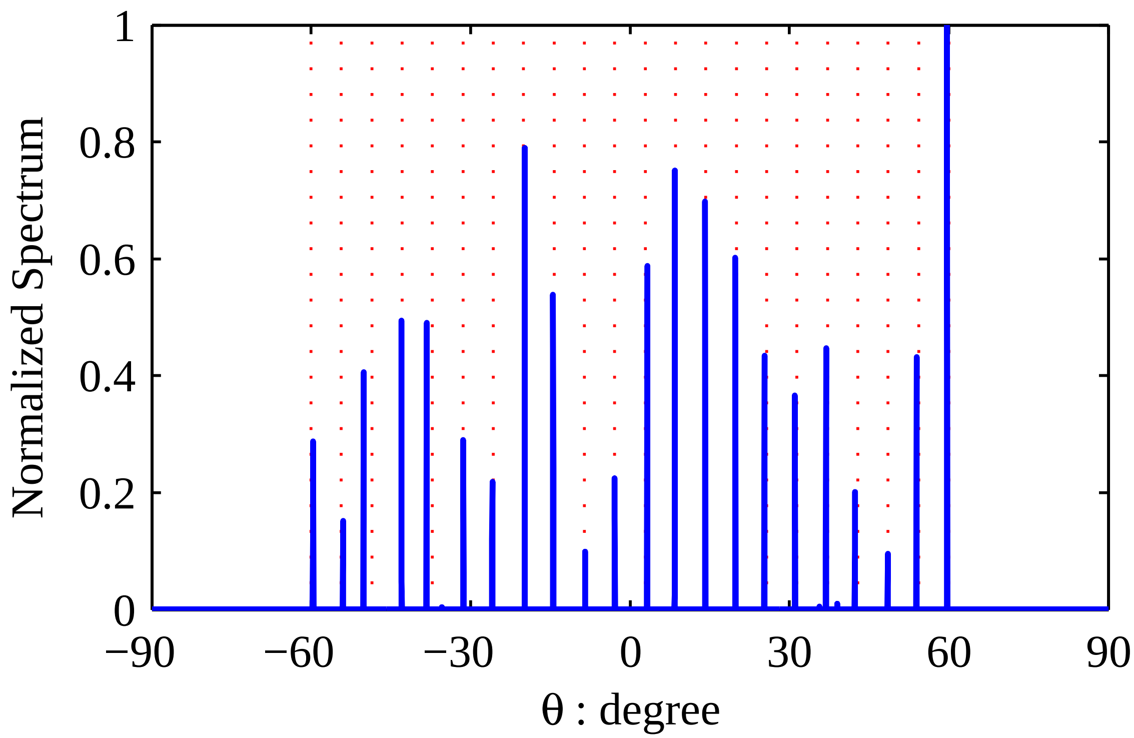
(a) DOA estimation results for the two-level nested array.



(b) DOA estimation results for the four-level nested array.

4. Simulation Results

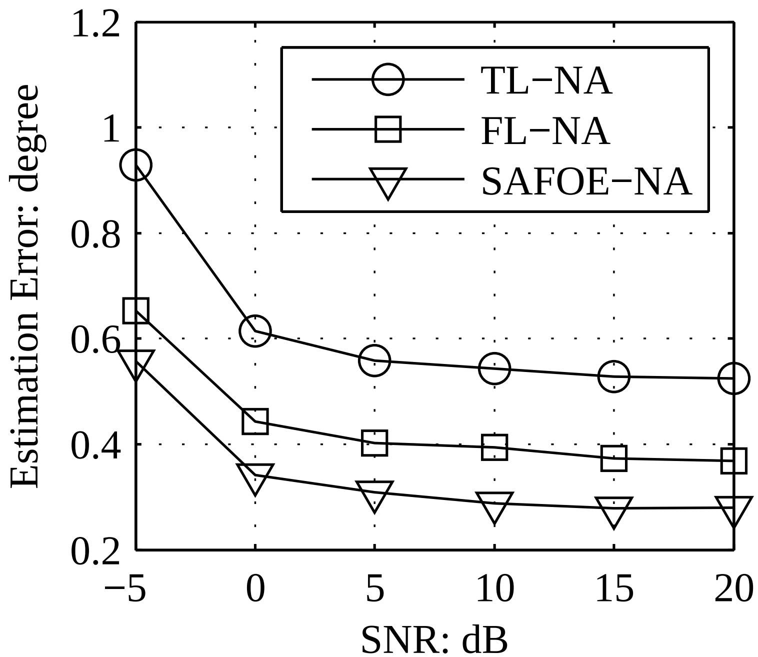
- DOA estimation results for our proposed structure (source number $K = 22$):



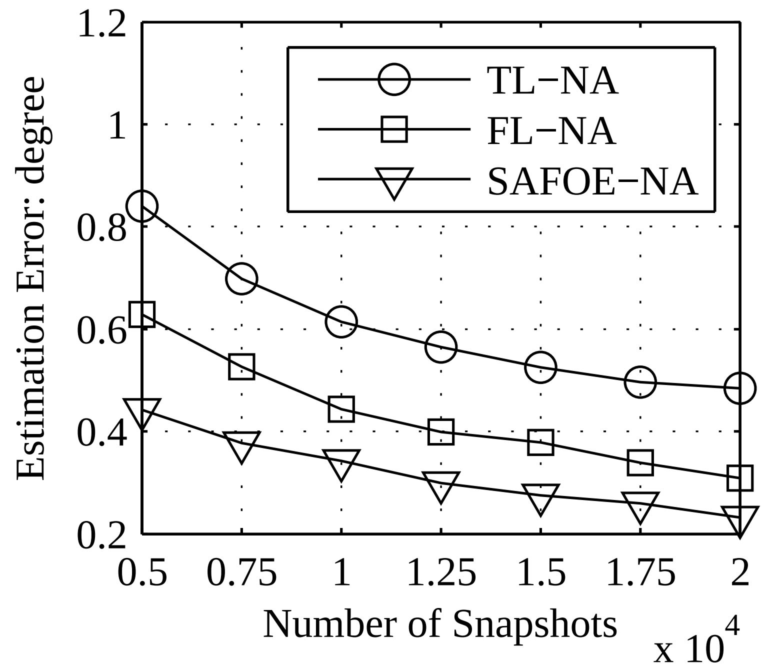
(c) DOA estimation results for our proposed structure.

4. Simulation Results

- RMSE results for different structures (source number $K = 12$):



(d) RMSEs versus input SNR.



(e) RMSEs versus the number of snapshots.

- The physical array aperture for the proposed structure is $23d$, while it is $15d$ for the FL-NA and $8d$ for the TL-NA.

5. Conclusion

- A sparse array extension based on the standard two-level nested array has been proposed to maximise the consecutive lags in the fourth-order difference co-array.
- After vectorizing the fourth-order cumulant matrix, a CS-based signal reconstruction method is then employed for effective DOA estimation.
- Given the same number of sensors, the number of consecutive lags and DOFs of the new structure is significantly larger than the two-level nested arrays; compared to the four-level nested arrays, when the sensor number is smaller than 21, the new structure also provides a larger number of consecutive lags (2223 for 20 sensors), which is sufficient for most applications.
- It can be shown that among the three different structures, the proposed one has the largest aperture, leading to further improved performance.

Thank you!