Distributed Network Filtering Based on Bayesian Quadratic Network Game

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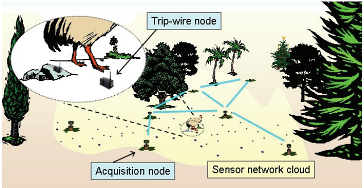
Outline

- Background
  - Network Signal Processing (NSP)
- Preliminary
  - Bayesian Network Game
  - Quadratic Network Game
  - Bayesian Quadratic Network Game (BQNG)
- Models
  - BQNG RF
- Simulation
- Conclusion

Background–Network Signal Processing (1/4)

- Sensor Network

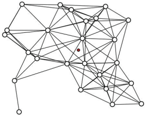


- Social Network

How a node process signals with interaction?

Background–Network Signal Processing (2/4)

- Estimation Problem



(a) Time-invarient

- Distributed Network Filter

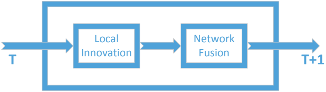


Figure: Structure of a Typical Distributed Network Filter

- Local Innovation

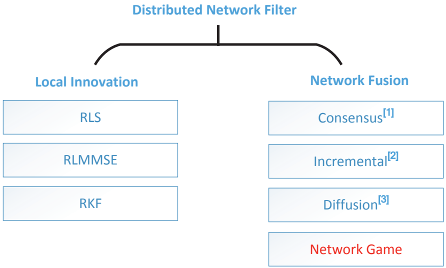
  update estimators using private information(self observation)

- Network Fusion

  update estimators using fusion information(neighbors information)

Background–Network Signal Processing (3/4)

- Progress in Research



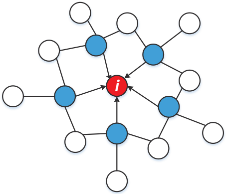
Why introduces *Network Game*?

[1] S. Kar and J. Moura, "Consensus + innovations distributed inference over networks: cooperation and sensing in networked systems", IEEE Signal Processing Mag., vol. 30, pp. 99-109, May 2013.

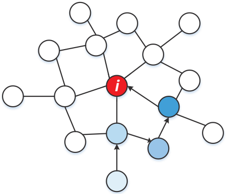
[2] H. Feng, Z. Jiang, B. Hu, and J. Zhang, "The incremental subgradient methods on distributed estimations in-network", SCI. CHINA Inf. Sci., vol. 57, no. 9, pp. 1-10, 2014.

[3] A. Sayed, "Adaptive networks", Proc. IEEE, vol. 102, no. 4, pp. 460-497, April 2014.

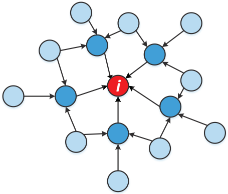
Background–Network Signal Processing (4/4)



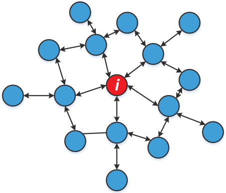
(a) Consensus



(b) Incremental



(c) Diffusion



(d) Network game

Figure: Information Quantity Comparison

Preliminary–Bayesian Network Game (1/1)

- A network game where the type of the other players are probabilistic.
- Definiton
  - A set of Player $\mathcal{V}$;
  - A graph topology $\mathcal{G}$;
  - A set of types of each player: $\theta \in \Theta_i$;
  - A set of actions (pure strategies) for each player: $a_i \in \mathcal{A}_i$;
  - A utility function of each player: $u_i(a_i, \mathbf{a}_{\text{-}i}, \theta_i, \mathcal{G}, \theta_{\text{-}i})$
- Action: $a_i : \Theta_i \mapsto \mathcal{A}_i$
- Bayesian Nash Equilibrium
  - A profile of pure strategy $\mathbf{a} = \{a_1, a_2, ..., a_N\}$ reaches to a stable state

$$\mathbb{E}_{\theta_{\text{-}i}}[u_i(a_i, \mathbf{a}_{\text{-}i}, \mathcal{G}, \theta_i, \theta_{\text{-}i})] \geq \mathbb{E}_{\theta_{\text{-}i}}[u_i(a_i^{'}, \mathbf{a}_{\text{-}i}, \mathcal{G}, \theta_i, \theta_{\text{-}i})]$$

for all $a_i^{'}$ in strategy space of i.

A Big Issue

Does any equilibrium exist?

Preliminary–Quadratic Network Game (1/1)

- Consider a game that each node in a network is implemented a quadratic utility as

$$u_{i,t}(a_{i,t}, a_{j,t}, \theta) = -\frac{1}{2}a_{i,t}^2 + \sum_{j\in\mathcal{V}\setminus i} \beta_{ij,t}a_{i,t}a_{j,t} + \delta a_{i,t}\theta$$

  - $a_{j,t}$: the action of node j at t
  - $\beta_{ij,t}$: the mutual effect intensity between node i and j
    - critical for the existence of equilibrium
    - critical for the interaction manner between nodes: strategy complimentary or strategy substitute
  - δ: the influence intensity of θ

A Big Issue

$a_{j,t}$ and θ may be unknown

Preliminary–BQNG* (1/5)

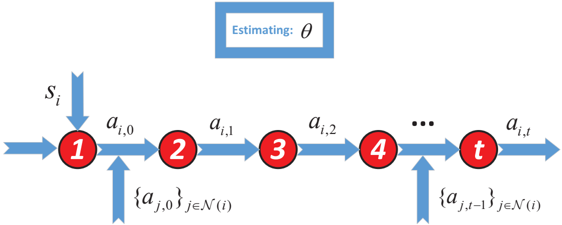
- Scenario

  Nodes in a network receive private observations of a random time-invariant variable θ only once as

  $$s_i = \theta + n_i, \quad \forall i \in \mathcal{V}, \quad n_i \sim \mathcal{N}(0, \sigma_i^2)$$

  Try to estimate θ under a sequetial Bayesian Network Game in a *distributed way*.

- Diagram



*Ceyhun Eksin, Pooya Molavi, Alejandro Ribeiro and Ali Jadbabaie. "Bayesian Quadratic Network Game Filters". IEEE Trans. Signal Processing. VOL. 62, NO. 9, MAY 1, 2014

Preliminary–BQNG (2/5)

- Game Models
  - player: nodes $i \in \mathcal{V} = \{1, 2, ..., N\}$
  - historical information: $h_{i,t} = \{h_{i,t\text{-}1}, a_{\mathcal{N}(i),t\text{-}1}\}, h_{i,0} = s_i$
  - Equilibia:

    $\sigma^*_{i,t} : h_{i,t} \mapsto a_{i,t}$

    History information are the types of players!
  - Best Strategy (Action) in equilibrium:

    Define $\mathbb{E}_{i,t}[\cdot] = \mathbb{E}_{h_{\text{-}i,t}}[\cdot|h_{i,t}]$, for any player i

    $$a^*_{i,t} = \sigma^*_{i,t}(h_{i,t}) = \arg\max_{a_{i,t} \in \mathbb{R}} \mathbb{E}_{i,t}[u_{i,t}(a_{i,t}, \mathbf{a}^*_{\text{-}i,t}, \theta)]$$

    $\Downarrow$ Zeroing the derivative of $u_{i,t}$ on $a_{i,t}$

    $$a^*_{i,t} = \sum_{j \in \mathcal{V} \setminus i} \beta_{ij,t} \mathbb{E}_{i,t}[a^*_{j,t}] + \delta \mathbb{E}_{i,t}[\theta]$$

Question

How to calculate the equilibrium?

- A Tractable Solution
  - The global observing vector (may not be a shared knowledge)

$$\mathbf{s} = [s_1, s_2, ..., s_N]^T$$

  - Linear form (Proved!)

$$\mathbb{E}_{i,t}[\mathbf{s}] = L_{i,t}\mathbf{s}, \quad \mathbb{E}_{i,t}[\theta] = k_{i,t}^T\mathbf{s}, \quad \mathbf{M}_{i,t} = \begin{bmatrix} M_{\theta\theta,i,t} & M_{\theta\mathbf{s},i,t} \\ M_{\mathbf{s}\theta,i,t} & M_{\mathbf{ss},i,t} \end{bmatrix}$$

  - Best strategy of linear form

$$a_{i,t} = v_{i,t}^T\mathbb{E}_{i,t}[\mathbf{s}] = v_{i,t}^T L_{i,t}\mathbf{s}$$

$$\mathbb{E}_{i,t}[a_{j,t}] = \mathbb{E}_{i,t}[v_{j,t}^T L_{j,t}\mathbf{s}] = v_{j,t}^T L_{j,t}\mathbb{E}_{i,t}[\mathbf{s}] = v_{j,t}^T L_{j,t} L_{i,t}\mathbf{s}$$

  - Rewrite the best response function

$$a_{i,t} = \sum_{j\in\mathcal{V}\backslash i} \beta_{ij,t}\mathbb{E}_{i,t}[a_{j,t}] + \delta\mathbb{E}_{i,t}[\theta]$$

$\Downarrow$ substitute these into the best response function

$$v_{i,t}^T L_{i,t}\mathbf{s} = \sum_{j\in\mathcal{V}\backslash i} \beta_{ij,t} v_{j,t}^T L_{j,t} L_{i,t}\mathbf{s} + \delta k_{i,t}^T\mathbf{s}$$

Preliminary–BQNG (4/5)

- The equilibrium solution

$$\begin{cases} \mathbf{L}_t &= \begin{bmatrix} L_{1,t}^T & \text{-}\beta_{12,t} L_{1,t}^T L_{2,t}^T & \cdots & \text{-}\beta_{1n,t} L_{1,t}^T L_{n,t}^T \\ \text{-}\beta_{21,t} L_{2,t}^T L_{1,t}^T & L_{2,t}^T & \cdots & \text{-}\beta_{2n,t} L_{2,t}^T L_{n,t}^T \\ \vdots & \cdots & \ddots & \vdots \\ \text{-}\beta_{n1,t} L_{n,t}^T L_{1,t}^T & \text{-}\beta_{n2,t} L_{n,t}^T L_{2,t}^T & \cdots & L_{n,t}^T \end{bmatrix} \\ \mathbf{k}_t &= [k_{1,t}^T, k_{1,t}^T, \ldots, k_{n,t}^T]^T \\ \mathbf{v}_t &= [v_{1,t}^T, v_{1,t}^T, \ldots, v_{n,t}^T]^T \end{cases}$$

$\Downarrow$ solving the system of equations

$$\mathbf{L}_t \mathbf{v}_t = \delta \mathbf{k}_t$$

- Calculate the action

$$a_{i,t} = v_{i,t}^T L_{i,t} \mathbf{s}$$

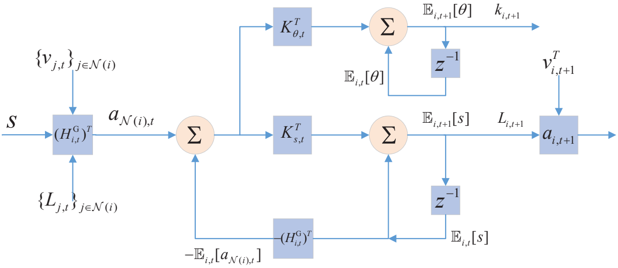
- Communications and update the estimators

$$H_{i,t}^{\mathrm{G}} = [L_{j_1,t\text{-}1}^T v_{j_1,t\text{-}1}, \ldots, L_{j_{d_i},t\text{-}1}^T v_{j_{d_i},t\text{-}1}] \in \mathbb{R}^{N \times d_i}$$

$$a_{\mathcal{N}(i),t} = (H_{i,t}^{\mathrm{G}})^T \mathbf{s}$$

Using $a_{\mathcal{N}(i),t}$ to update $L_{i,t}$, $k_{i,t}$, $\mathbf{M}_{i,t}$

- Update of estimators



- What's the problem for a realization of distributed network filter?
  - Multiple observations: $\mathbf{s} \rightarrow \mathbf{s}_t$
  - Time-varying state: $\theta \rightarrow \theta_t$

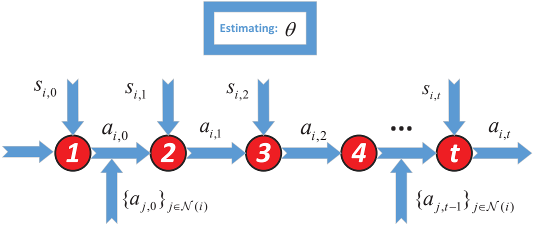
Models–BQNG RF (1/7)

- Scenario

  Nodes in a WSNs receive private observations as

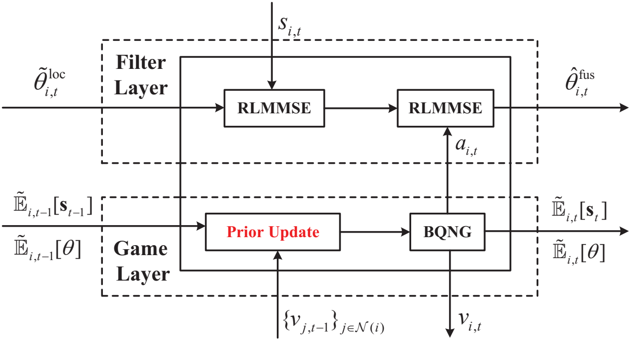
$$s_{i,t} = \theta + n_{i,t}, \quad \forall i \in \mathcal{V}, \quad n_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$$

- Diagram



Models–BQNG RF (2/7)

- Structure



- Filter Layer

  Updates the estimation by local and fusion information

- Game Layer

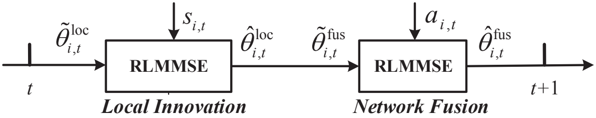
  Deals with the information fusion by the BQNG

Models–BQNG RF (3/7)

- Filter layer

  Updates the estimation by local and fusion information

  - Structure



  - Using the private observation $s_{i,t}$ update $\hat{\theta}$

$$s_{i,t} = \theta + n_{i,t}$$

Then implement RLMMSE

$$\begin{cases} K^{\text{loc}}_{i,t} &= \hat{Q}^{\text{fus}}_{\theta\theta,i,t\text{-}1} H^{\text{loc}}_{i,t} (\sigma^2 + (H^{\text{loc}}_{i,t})^T \hat{Q}^{\text{fus}}_{\theta\theta,i,t\text{-}1} H^{\text{loc}}_{i,t})^{-1} \\ \hat{\theta}^{\text{loc}}_{i,t} &= \hat{\theta}^{\text{fus}}_{i,t\text{-}1} + K^{\text{loc}}_{i,t}(s_{i,t} - \hat{\theta}^{\text{fus}}_{i,t\text{-}1}) \\ \hat{Q}^{\text{loc}}_{\theta\theta,i,t} &= (I - K^{\text{loc}}_{i,t}(H^{\text{loc}}_{i,t})^T)\hat{Q}^{\text{fus}}_{\theta\theta,i,t\text{-}1} \end{cases}$$

Models–BQNG RF (4/7)

- Using the fusion information $a_{i,t}$ update $\hat{\theta}$

$$a_{i,t} = H_{i,t}^{\text{fus}}\theta + n_t^{\text{fus}}$$

Then implement RLMMSE

$$\begin{cases} K_{i,t}^{\text{fus}} &= \tilde{Q}_{\theta\theta,i,t}^{\text{fus}} H_{i,t}^{\text{fus}} (\sigma_t^{\text{fus}} + (H_{i,t}^{\text{fus}})^T \tilde{Q}_{\theta\theta,i,t}^{\text{fus}} H_{i,t}^{\text{fus}})^{-1} \\ \hat{\theta}_{i,t}^{\text{fus}} &= \tilde{\theta}_{i,t}^{\text{fus}} + K_{i,t}^{\text{fus}} (a_{i,t} - \tilde{\theta}_{i,t}^{\text{fus}}) \\ \hat{Q}_{\theta\theta,i,t}^{\text{fus}} &= (I - K_{i,t}^{\text{fus}} (H_{i,t}^{\text{fus}})^T) \tilde{Q}_{\theta\theta,i,t}^{\text{fus}} \end{cases}$$

Theorem 1

$\hat{\theta}_{i,t}^{\text{loc}}$ and $\hat{\theta}_{i,t}^{\text{fus}}$ are unbiased estimations on θ at any t.

Proof.

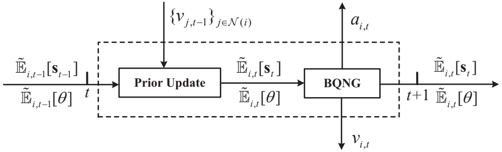
We prove by induction. At initialization, $\tilde{\theta}_{i,0}^{\text{loc}} = s_{i,0}$, which is unbiased estimation upon θ. Since RLMMSE remains the unbiasedness of an estimator under the Gaussian Model, **Theorem 1** is proved. □

Models–BQNG RF (5/7)

- Game Layer

  Deals with the information fusion by the BQNG

  - Structure



  - Prior Update

$$\tilde{\mathbb{E}}_{i,t\text{-}1}[\mathbf{s}_{t\text{-}1}] \sim \mathcal{N}(\tilde{L}_{i,t\text{-}1}\mathbf{s}_{t\text{-}1}, \tilde{M}_{ss,i,t\text{-}1}),\ \tilde{\mathbb{E}}_{i,t\text{-}1}[\theta] \sim \mathcal{N}(\tilde{k}_{i,t\text{-}1}^{\mathrm{T}}\mathbf{s}_{t\text{-}1}, \tilde{M}_{\theta\theta,i,t\text{-}1})$$

$\Downarrow$ estimator prediction

$$\hat{\mathbb{E}}_{i,t}[\mathbf{s}_{t\text{-}1}] \sim \mathcal{N}(\hat{L}_{i,t\text{-}1}\mathbf{s}_{t\text{-}1}, \hat{M}_{ss,i,t\text{-}1}),\ \hat{\mathbb{E}}_{i,t\text{-}1}[\theta] \sim \mathcal{N}(\hat{k}_{i,t\text{-}1}^{T}\mathbf{s}_{t\text{-}1}, \hat{M}_{\theta\theta,i,t\text{-}1})$$

$\Downarrow$ global vector prediction

$$\tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t] \sim \mathcal{N}(\tilde{L}_{i,t}\mathbf{s}_t, \tilde{M}_{ss,i,t}),\ \tilde{\mathbb{E}}_{i,t}[\theta] \sim \mathcal{N}(\tilde{k}_{i,t}^{\mathrm{T}}\mathbf{s}_t, \tilde{M}_{\theta\theta,i,t})$$

Model–BQNG RF (6/7)

- Estimator Prediction: $\tilde{\mathbb{E}}_{i,t\text{-}1}[\mathbf{s}_{t\text{-}1}] \to \hat{\mathbb{E}}_{i,t}[\mathbf{s}_{t\text{-}1}]$, $\tilde{\mathbb{E}}_{i,t\text{-}1}[\theta] \to \hat{\mathbb{E}}_{i,t\text{-}1}[\theta]$

- Observing Function

$$H^{\mathrm{G}}_{i,t} = [L^T_{j_{i,1},t\text{-}1} v_{j_{i,1},t\text{-}1}, \ldots, L^T_{j_{i,d_i},t\text{-}1} v_{j_{i,d_i},t\text{-}1}] \in \mathbb{R}^{N \times d_i}$$

$$\mathbf{S}_{a_{\mathcal{N}(i)},t} = (H^{\mathrm{G}}_{i,t})^T \mathbf{s}_{t\text{-}1}$$

- Prediction by RLMMSE

$$\begin{aligned}
G_{\theta,i,t} &= M_{\theta \mathbf{s}_{t\text{-}1},i,t\text{-}1} H^{\mathrm{G}}_{i,t} ((H^{\mathrm{G}}_{i,t})^T M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t\text{-}1} H^{\mathrm{G}}_{i,t})^{-1}, \\
G_{\mathbf{s}_{t\text{-}1},i,t} &= M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t\text{-}1} H^{\mathrm{G}}_{i,t} ((H^{\mathrm{G}}_{i,t})^T M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t\text{-}1} H^{\mathrm{G}}_{i,t})^{-1}, \\
k^T_{i,t} &= k^T_{i,t\text{-}1} + G_{\theta,i,t}((H^{\mathrm{G}}_{i,t})^T - (H^{\mathrm{G}}_{i,t})^T L_{i,t\text{-}1}), \\
L_{i,t} &= L_{i,t\text{-}1} + G_{\mathbf{s}_{t\text{-}1},i,t}((H^{\mathrm{G}}_{i,t})^T - (H^{\mathrm{G}}_{i,t})^T L_{i,t\text{-}1}), \\
M_{\theta\theta,i,t} &= M_{\theta\theta,i,t\text{-}1} - G_{\theta,i,t}(H^{\mathrm{G}}_{i,t})^T M_{\theta\theta,i,t\text{-}1}, \\
M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t} &= M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t\text{-}1} - G_{\mathbf{s}_{t\text{-}1},i,t}(H^{\mathrm{G}}_{i,t})^T M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t\text{-}1}, \\
M_{\theta \mathbf{s}_{t\text{-}1},i,t} &= M_{\theta \mathbf{s}_{t\text{-}1},i,t\text{-}1} - G_{\theta_{t\text{-}1},i,t}(H^{\mathrm{G}}_{i,t})^T M_{\mathbf{s}_{t\text{-}1}\mathbf{s}_{t\text{-}1},i,t\text{-}1}.
\end{aligned}$$

Models–BQNG RF (7/7)

- global Vector Prediction: $\hat{\mathbb{E}}_{i,t}[\mathbf{s}_{t\text{-}1}] \rightarrow \tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t]$, $\hat{\mathbb{E}}_{i,t\text{-}1}[\theta] \rightarrow \tilde{\mathbb{E}}_{i,t}[\theta]$

- $\mathbf{s}_t = \mathbf{s}_{t\text{-}1} + w_t$,
- $\mathbb{E}_{i,t}[\mathbf{s}_t] = \mathbb{E}_{i,t}[\mathbf{s}_{t\text{-}1}] + \mathbf{w}_t$
- $\mathbb{E}_{i,t}[\mathbf{s}_t] = \hat{L}_{i,t\text{-}1}\mathbf{s}_{t\text{-}1} + \mathbf{m}_t + \mathbf{w}_t$

$\Downarrow$ substitute with $\mathbf{s}_{t\text{-}1} = \mathbf{s}_t + w_t$

$$\mathbb{E}_{i,t}[\mathbf{s}_t] = \hat{L}_{i,t\text{-}1}\mathbf{s}_t + \mathbf{m}_t + \mathbf{w}_t + \mathbf{r}_t$$

$\Downarrow$ $\tilde{L}_{i,t} = \hat{L}_{i,t\text{-}1}$, $\tilde{M}_{ss,i,t} = \mathbf{m}_t + \mathbf{w}_t + \mathbf{r}_t$

$$\mathbb{E}_{i,t}[\mathbf{s}_t] = \tilde{L}_{i,t}\mathbf{s}_t + \tilde{M}_{ss,i,t}$$

Theorem 2

In the game layer, structure as $\tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t] \sim \mathcal{N}(\tilde{L}_{i,t}\mathbf{s}_t, \tilde{M}_{ss,i,t})$, $\tilde{\mathbb{E}}_{i,t}[\theta] \sim \mathcal{N}(\tilde{k}_{i,t}^{\mathrm{T}}\mathbf{s}_t, \tilde{M}_{\theta\theta,i,t})$ *always hold with initialization*
$\tilde{\mathbb{E}}_{i,0}[\mathbf{s}_0] \sim \mathcal{N}(\tilde{L}_{i,0}\mathbf{s}_0, \tilde{M}_{ss,i,0}), \tilde{\mathbb{E}}_{i,0}[\theta] \sim \mathcal{N}(\tilde{k}_{i,0}^{\mathrm{T}}\mathbf{s}_0, \tilde{M}_{\theta\theta,i,0})$.

Simulation–BQNG RF (1/2)

- Initialization

$$\tilde{\mathbb{E}}_{i,0}[\mathbf{s}_0] \sim \mathcal{N}(\tilde{L}_{i,0}\mathbf{s}_0, \tilde{M}_{ss,i,0})$$

$$\tilde{\mathbb{E}}_{i,0}[\theta] \sim \mathcal{N}(\tilde{k}_{i,0}^{\mathrm{T}}\mathbf{s}_0, \tilde{M}_{\theta\theta,i,0})$$

where $\tilde{L}_{i,0} = \mathbf{1}\mathbf{e}_i^{\mathrm{T}}$, $\tilde{k}_{i,0}^{T} = \mathbf{e}_i^{\mathrm{T}}$, $\tilde{M}_{\theta\theta,i,0} = \sigma_i^2$, $\tilde{M}_{\theta s,i,0} = \sigma_i^2\overline{\mathbf{e}_i}^{\mathrm{T}}$ and $\tilde{M}_{ss,i,0} = \sigma_{i,0}^2(\mathbf{1}\overline{\mathbf{e}_i}^{\mathrm{T}} + \overline{\mathbf{e}_i}\overline{\mathbf{e}_i}^{\mathrm{T}})$. $\mathbf{e}_i \in \mathbb{R}^n$ is the vector where 1 in i'th column and 0 otherwise, and $\overline{\mathbf{e}_i} = \mathbf{1} - \mathbf{e}_i$.

- Parameters
  - Target:

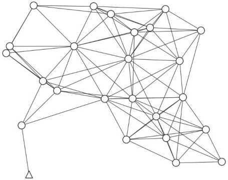
    $\theta = 5$
  - Observing function:

    $s_{i,t} = \theta + n_{i,t}, \quad \forall i \in \mathcal{V}, \quad n_{i,t} \sim \mathcal{N}(0,1)$
  - WSNs:

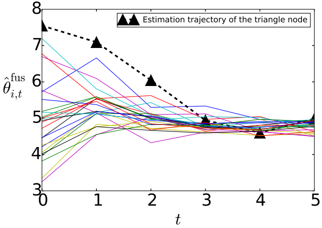
    25 nodes, random topology

Simulation–BQNG RF (2/2)

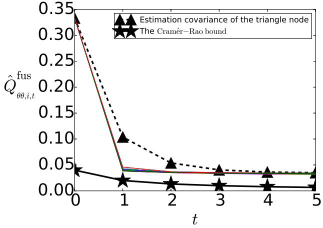
- WSNs topology



- Estimating Performances



(a) Estimations of WSNs.



(b) Est variances of WSNs.

Conclusion (1/1)

- The BQNG contains more global information than traditional methods.
- The two-layer BQNG framework could be generalized to more filtering scenarios.
- Future work
  - In view of the network part

    Communicational complexity, robust to varying network topology
  - In view of the game part

    Analyzing the existence and structure of the equilibrium

Q&A