Rate Analysis for Detection of Sparse Mixtures

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Detecting Sparse Mixtures

- Test between pure noise and sparse signal in noise
- Sparse signal in noise modeled as mixture between noise and signal PDF
- Study trade-off between signal strength, sparsity and sample size
- Applications: Sensor Networks, Disease Outbreak Monitoring, Astrophysics, Bioinformatics, Covert Signaling [2,8-11]
- Initially studied with unit variance Gaussian noise and signal
- Three Questions:
    1. When are there consistent tests?
    2. What are the best rates for consistent tests?
    3. Are there adaptive tests (i.e. unknown signal and noise) with best rate?

Mathematical Model

- Test between:

$$H_{0,n}: X_1, \dots X_n \sim f_{0,n}(x)$$
$$H_{1,n}: X_1, \dots X_n \sim (1-\epsilon_n) f_{0,n}(x) + \epsilon_n f_{1,n}(x)$$

- $\{f_{0,n}(x)\}, \{f_{1,n}(x)\}$ sequence of PDFs; $L_n(x) = \frac{f_{1,n}(x)}{f_{0,n}(x)}$
- $\epsilon_n \to 0, n\,\epsilon_n \to \infty$
- $\mathrm{LLR}(n) = \sum_{i=1}^{n} \log\left(1 - \epsilon_n + \epsilon_n L_n(x_i)\right)$
- Analyze rate and consistency of oracle LRT:
- $\delta_n(x_1, \dots, x_n) = \begin{cases} 1 & \mathrm{LLR}(n) \geq 0 \\ 0 & \mathrm{LLR}(n) < 0 \end{cases}$
- False Alarm: $\mathrm{P}_{\mathrm{FA}}(n) = \mathrm{P}_0[\delta_n = 1]$
- Miss Detection: $\mathrm{P}_{\mathrm{MD}}(n) = \mathrm{P}_1[\delta_n = 0]$
- Consistency: $\mathrm{P}_{\mathrm{FA}}(n), \mathrm{P}_{\mathrm{MD}}(n) \to 0$ as $n \to \infty$
- Prior work [2,3,4,5,6] focuses on consistency and adaptivity, not rate

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Rate Analysis For the LRT

- Rates of consistency: How quickly do error probabilities tend to zero?
- Rate Characterization: $\lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{FA}}(n)}{g_0(n)} = -c.\ \lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{MD}}(n)}{g_1(n)} = -d$

  where $c, d > 0$ and $g_0(n), g_1(n) \to \infty$
- Classic i.i.d. Hypothesis Testing: $g_0(n) = g_1(n) = n$, error exponents (KL divergence); Sublinear $g_0(n), g_1(n)$ in this work
- Rate Characterization for LRT:

Weak Signals: Assume that for all $0 < \gamma < \gamma_0$ where $\gamma_0 \in (0,1)$:

1. $\lim_{n\to\infty} \mathrm{E}_0\left[\frac{(L_n-1)^2}{D_n^2}; L_n \geq 1 + \frac{\gamma}{\epsilon_n}\right] = 0$
2. $\epsilon_n D_n \to 0, n\,\epsilon_n^2 D_n^2 \to \infty$

Where $D_n^2 = \mathrm{E}_0[(L_n - 1)^2] < \infty$ is the χ^2-divergence between $f_{0,n}, f_{1,n}$.

Then, $\lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{FA}}(n)}{n\epsilon_n^2 D_n^2} = \lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{MD}}(n)}{n\epsilon_n^2 D_n^2} = -\frac{1}{8}$

Strong Signals: If for all M sufficiently large, $\lim_{n\to\infty} \mathrm{E}_0\left[L_n; L_n \geq 1 + \frac{M}{\epsilon_n}\right] = 1$,

then $\lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{FA}}(n)}{n\epsilon_n} \leq -1,\ \lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{MD}}(n)}{n\epsilon_n} = -1$

- Weak signals characterized by χ^2-divergence
    - Proof: Chernoff (UB), Mod. Cramer's Theorem w. n-dependent tilting (LB)
- Strong signal rate is independent of divergence (beyond condition)
    - Proof: Chernoff (UB), Universal $\liminf_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{MD}}(n)}{n\epsilon_n} \geq -1$ (typicality LB)

Gaussian Location Model (GLM)

- $f_{0,n} = \mathcal{N}(0,1),\ f_{1,n} = \mathcal{N}(\mu_n, 1)$ and $\epsilon_n = n^{-\beta}, \beta \in (0,1)$
- $\{(\epsilon_n, \mu_n)\}$ conditions for consistency [2,3,4,5]
- Adaptive testing (without rate guarantees) [2,3,4,5,6] (Different tests have very different power [4,7])

Conditions for consistency [2,3,4,5]:

1. (Dense) If $0 < \beta < \frac{1}{2}, \mu_{crit,n} = n^{\beta - \frac{1}{2}}$
2. (Moderately Sparse) If $\frac{1}{2} < \beta < \frac{3}{4}, \mu_{crit,n} = \sqrt{2(\beta - \frac{1}{2}) \log n}$
3. (Very Sparse) If $\frac{3}{4} < \beta < 1, \mu_{crit,n} = \sqrt{2\left(1 - \sqrt{1-\beta}\right)^2 \log n}$

If $\mu_n < \mu_{crit,n}, P_{FA} + P_{MD} \to 1$ (Detection is impossible)

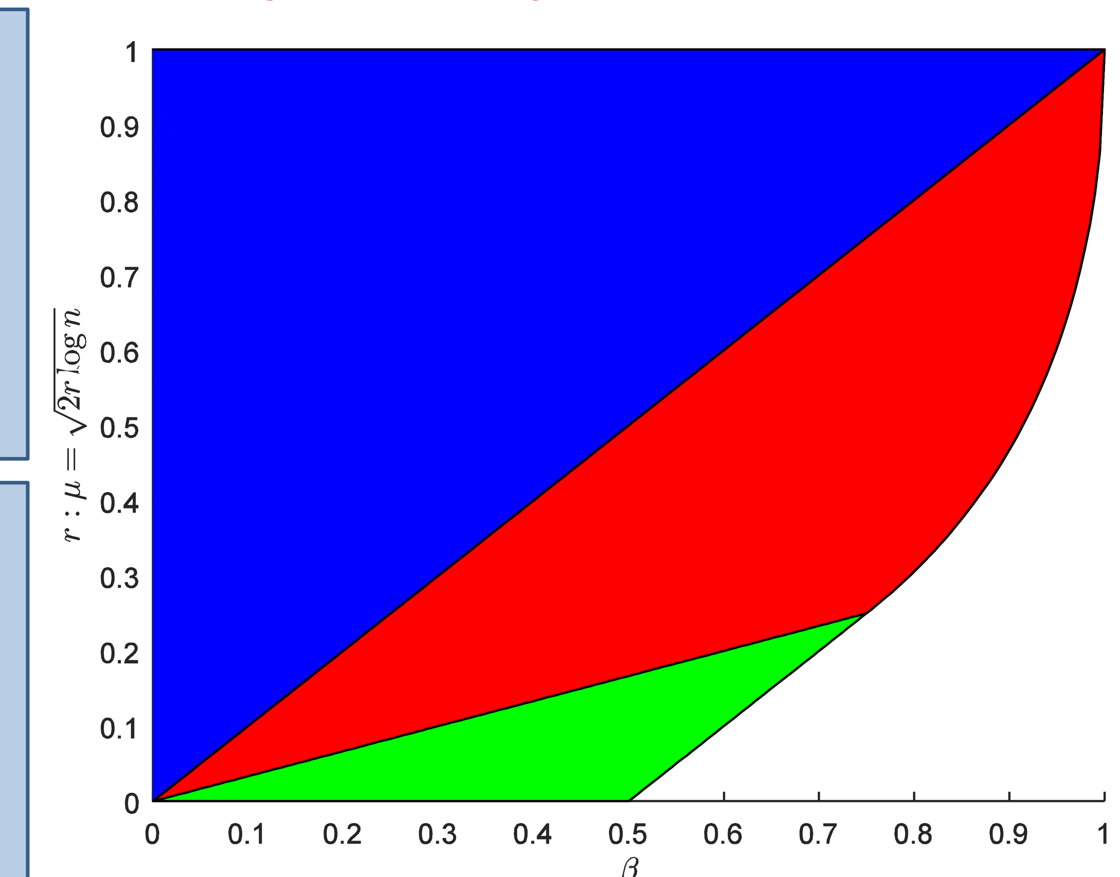
Rate Characterization for GLM:

Weak Signals (Green): If $\mu_{crit,n} < \mu_n < \sqrt{\frac{2\beta}{3} \log n}$,

$\lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{FA}}(n)}{n\epsilon_n^2 (e^{\mu_n^2} - 1)} = \lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{MD}}(n)}{n\epsilon_n^2 (e^{\mu_n^2} - 1)} = -\frac{1}{8}$

Strong Signals (Blue): If $\mu_n > \sqrt{2\beta \log n}$, $\lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{MD}}(n)}{n\epsilon_n} = -1.$

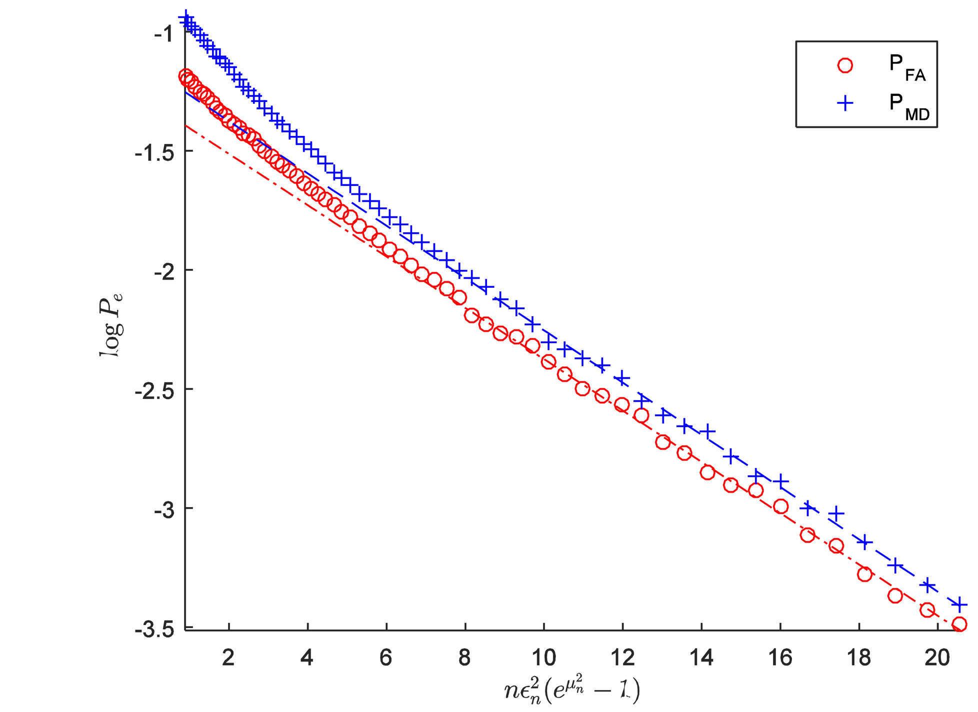
Furthermore, if $\frac{n\epsilon_n}{\mu_n^2} \to \infty$, $\lim_{n\to\infty} \frac{\log \mathrm{P}_{\mathrm{FA}}(n)}{n\epsilon_n} = -1$



- LB for $\mathrm{P}_{\mathrm{FA}}(\mathrm{n})$ via LB on $\mathrm{LLR}(n)$ + Modified Cramer's Theorem

Numerical Results (GLM)

- $\epsilon_n = n^{-0.6}, \mu_n = \sqrt{2(0.19)\log n}$ (Sparse, Weak)
- Best fit slope with $n \geq 100000$: -0.108 (FA, MD)
- Theoretical prediction: $-\frac{1}{8}$



Conclusions and Future Work

- Analyzed rates for oracle LRT for general sparse mixtures
- Sublinear rate with non-KL divergence
- Future Work: Design adaptive tests with oracle rate for GLM and other mixture models

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