Fast and Statistically Efficient Fundamental Frequency Estimation

ICASSP 2016

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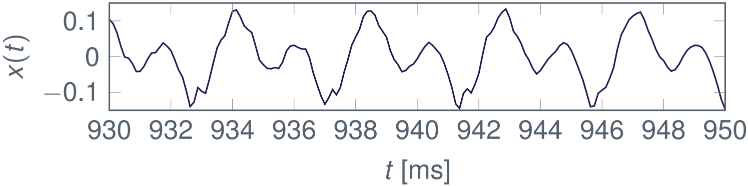
DENMARK

Agenda

Speech Signal Example



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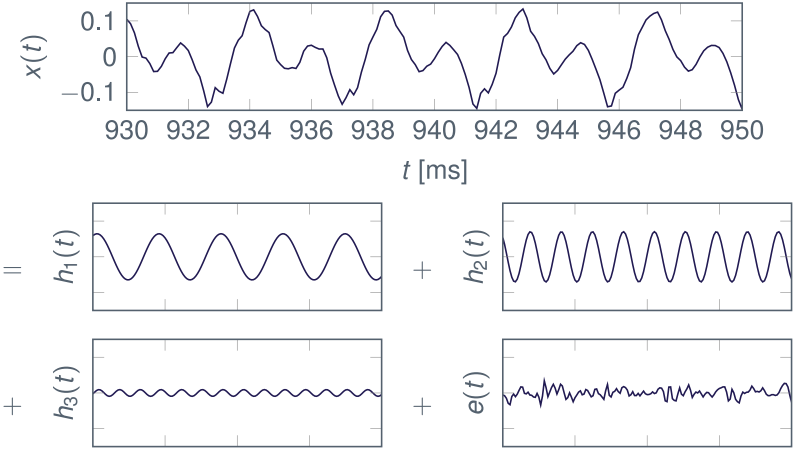

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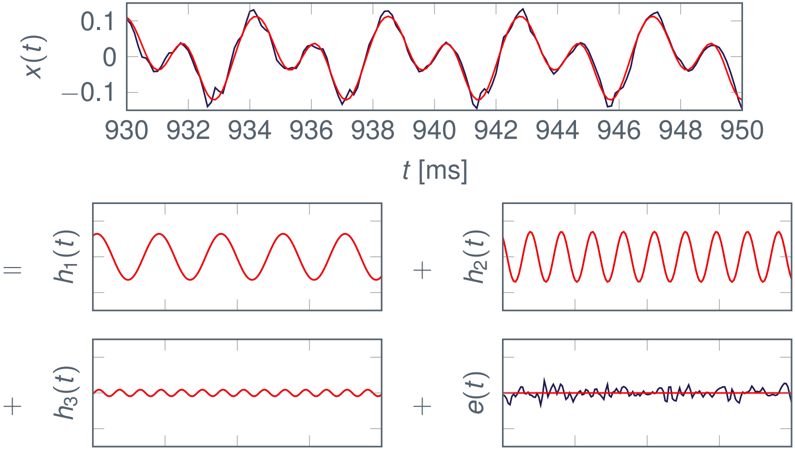

Mathematical Model

Mathematical Model

$$x(t) = \sum_{i=1}^{l} h_i(t) + e(t) = \sum_{i=1}^{l} A_i \cos(i2\pi f_0 t + \phi_i) + e(t) \qquad (1)$$

where

- A_i real amplitude of the *i*th harmonic
- ϕ_i phase of the *i*th harmonic
- f_0 fundamental frequency in Hz
- l the number of harmonics/model order
- $e(t)$ white Gaussian noise with variance σ^2

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Analysis problem: Get f_0 and l from the data.

Estimation Methods

Correlation Methods

A periodic signal satisfies that

$$x(t) = x(t + T) \tag{2}$$

where $T = 1/f_0$ is the period. Thus, the autocorrelation function of $x(t)$ has a peak for a lag of T.

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+ Low computational complexity
+ No need to estimate the model order
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Correlation methods such as YIN and RAPT are very popular.

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Parametric Methods

Estimate the parameters in

$$x(t) = \sum_{i=1}^{l} A_i \cos(i2\pi f_0 t + \phi_i) + e(t) \tag{3}$$

$$= \sum_{i=1}^{l} \left[a_i \cos(i2\pi f_0 t) - b_i \sin(i2\pi f_0 t) \right] + e(t) \tag{4}$$

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- \+ Work very well in even noisy conditions
- \+ Work for low fundamental frequencies
- \- The model order has to be estimated
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- \- High computational complexity (for MLE/NLS)

Periodic Signals
Vector Signal Model

The sampled signal model

$$x(n) = \sum_{i=1}^{l} \Big[a_i \cos(i\omega_0 n) - b_i \sin(i\omega_0 n) \Big] + e(n) \tag{5}$$

for $n = n_0, n_0 + 1, \ldots, n_0 + N - 1$ can be written as

$$\boldsymbol{x} = \boldsymbol{Z}_l(\omega_0)\boldsymbol{\alpha}_l + \boldsymbol{e}\,. \tag{6}$$

where

$$\boldsymbol{Z}_l(\omega) = \begin{bmatrix} \boldsymbol{c}(\omega) & \boldsymbol{c}(2\omega) & \cdots & \boldsymbol{c}(l\omega) & \boldsymbol{s}(\omega) & \boldsymbol{s}(2\omega) & \cdots & \boldsymbol{s}(l\omega) \end{bmatrix}$$

$$\boldsymbol{c}(\omega) = \begin{bmatrix} \cos(\omega n_0) & \cdots & \cos(\omega(n_0 + N - 1)) \end{bmatrix}^T$$

$$\boldsymbol{s}(\omega) = \begin{bmatrix} \sin(\omega n_0) & \cdots & \sin(\omega(n_0 + N - 1)) \end{bmatrix}^T$$

$$\boldsymbol{\alpha}_l = \begin{bmatrix} \boldsymbol{a}_l^T & -\boldsymbol{b}_l^T \end{bmatrix}^T, \ \boldsymbol{a}_l = \begin{bmatrix} a_1 & \cdots & a_l \end{bmatrix}^T, \ \boldsymbol{b}_l = \begin{bmatrix} b_1 & \cdots & b_l \end{bmatrix}^T.$$

Maximum Likelihood Estimation

The maximum likelihood estimate (MLE) for the fundamental frequency is

$$\hat{\omega}_0 = \underset{\omega_0}{\operatorname{argmax}}\, \boldsymbol{x}^T \boldsymbol{Z}_l(\omega_0) \left[\boldsymbol{Z}_l^T(\omega_0)\boldsymbol{Z}_l(\omega_0)\right]^{-1} \boldsymbol{Z}_l^T(\omega_0)\boldsymbol{x} \tag{7}$$

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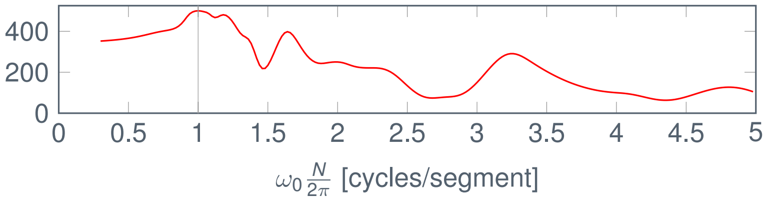
- The ML/NLS estimator has been known since Quinn and Thomson (1991), but is costly to compute.

Maximum Likelihood Estimation



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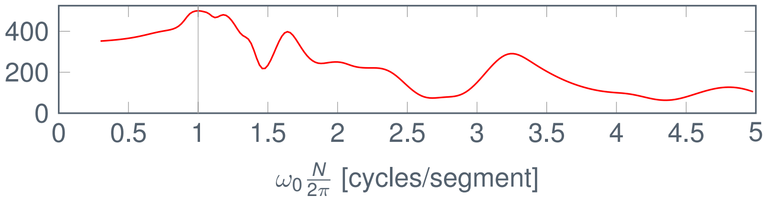

1. Compute NLS cost function

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on an F/l-point uniform grid for all model orders $l \in \{1, \ldots, L\}$.

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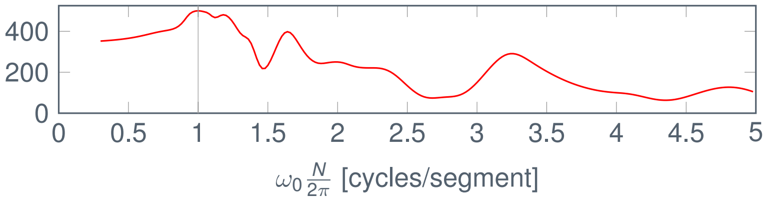

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2. Optionally refine the L grid estimates.

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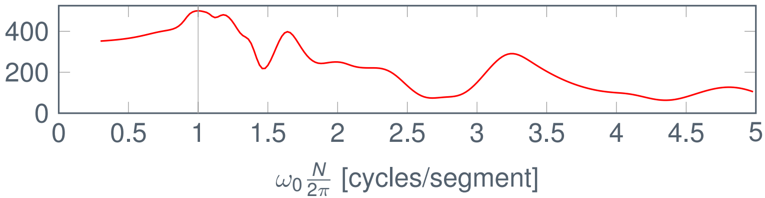

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3. Do model comparison.

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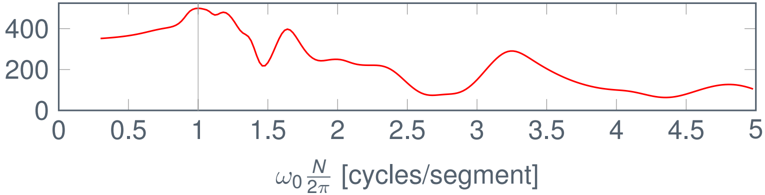

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The harmonic summation (HS) estimator

$$\hat{\omega}_0 \approx \underset{\omega_0 \in (0, \pi / l)}{\operatorname{argmax}} \ \boldsymbol{x}^T \boldsymbol{Z}_l(\omega_0) \left[N \boldsymbol{I}_l / 2 \right]^{-1} \boldsymbol{Z}_l^T(\omega_0) \boldsymbol{x} \ . \qquad (9)$$

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ML/NLS	$\mathcal{O}(F \log F) + \mathcal{O}(FL^3)$
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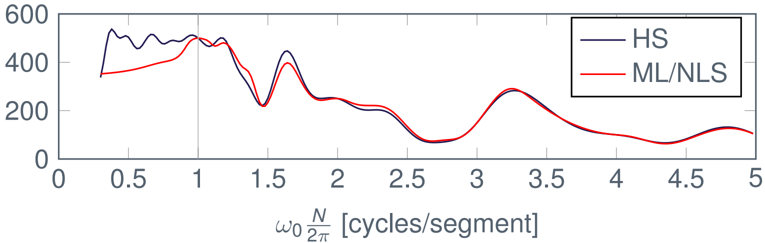
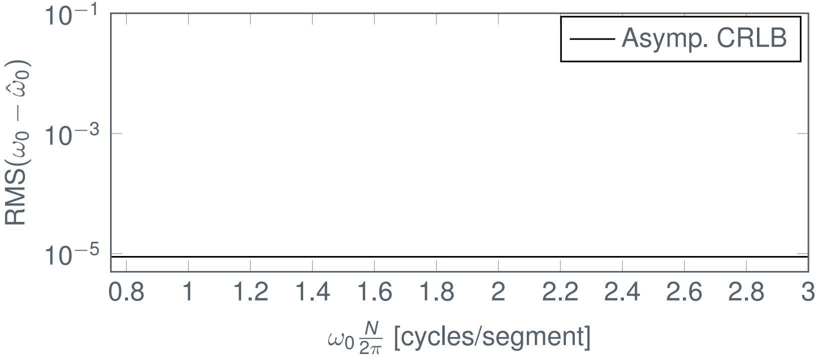


Estimation accuracy for a high fundamental frequency

Setup: $N = 200$, $l = 10$, 10000 repetitions, random phases, and constant amplitudes, and SNR of 15 dB



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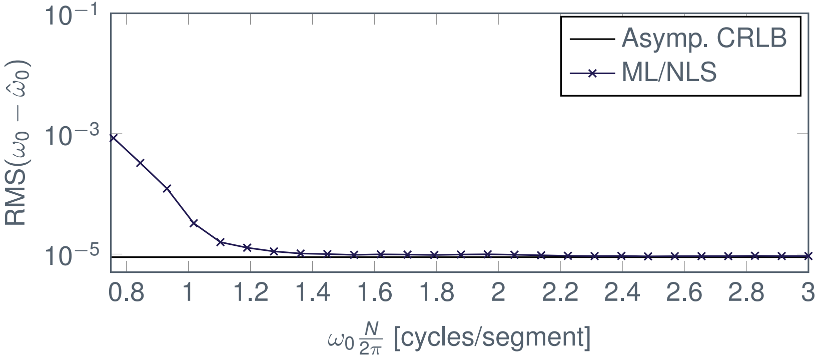

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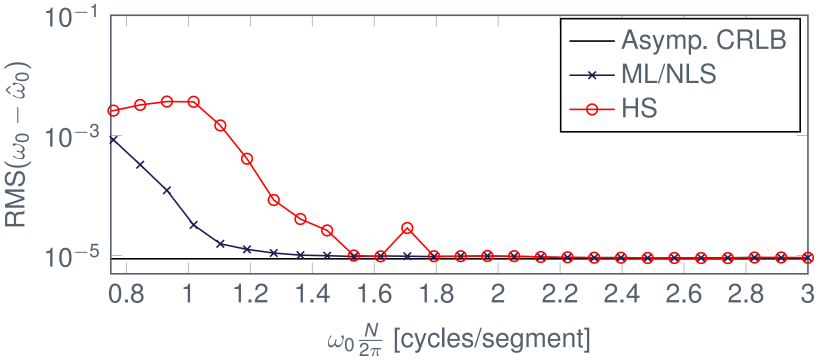

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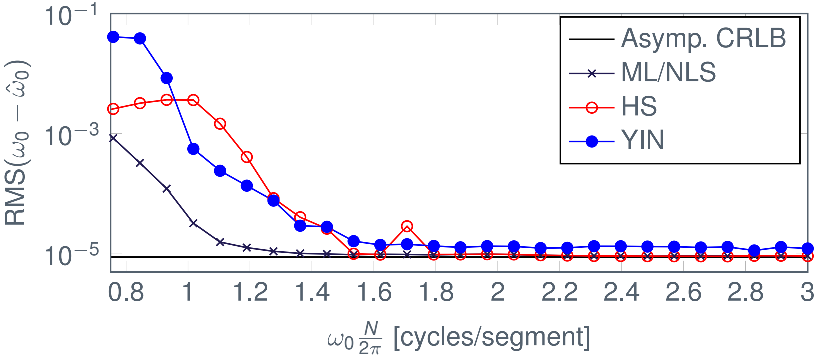

Estimation accuracy for a low fundamental frequency

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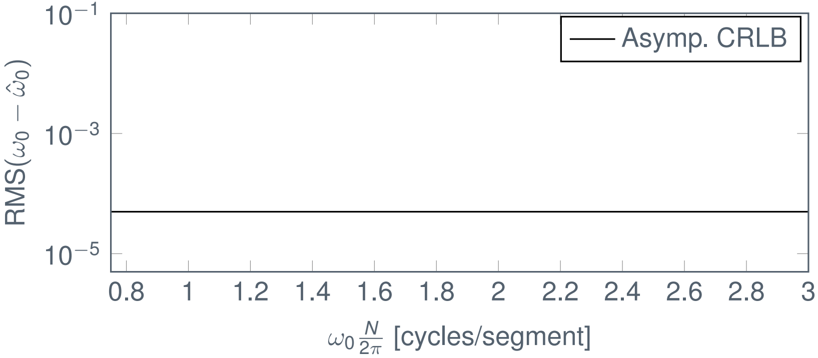

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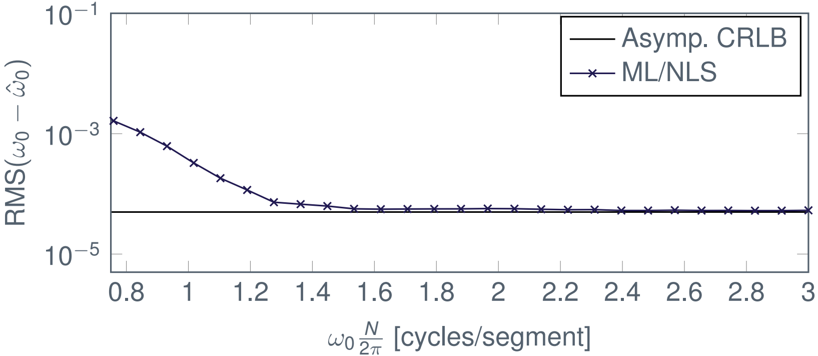

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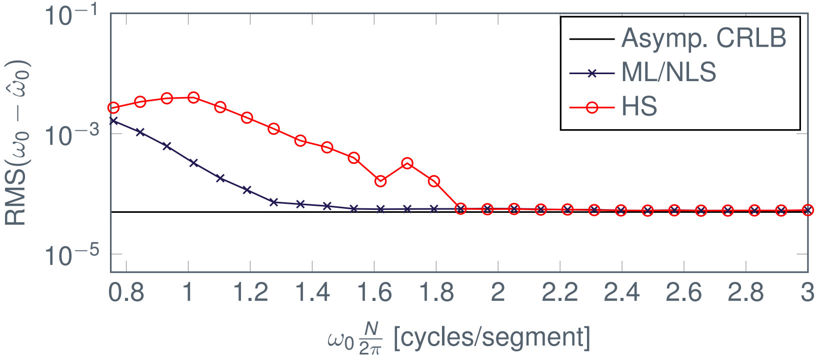

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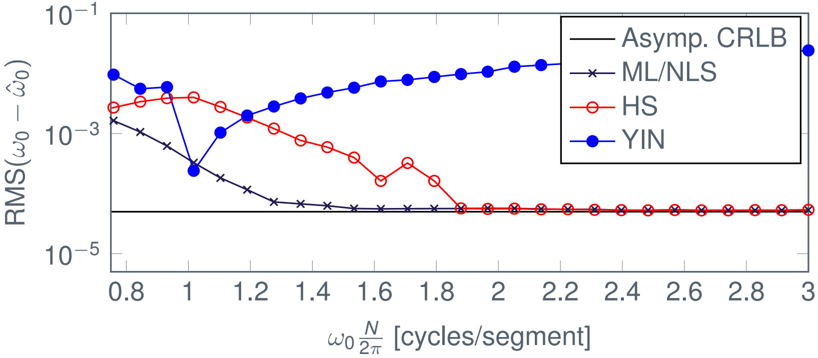

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Contribution

ML/NLS: $\mathcal{O}(F \log F) + \mathcal{O}(FL)$

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Fundamental Frequency Estimation

Maximum Likelihood Estimation

Fast Maximum Likelihood Estimation

Results

Summary

Fast ML/NLS

$$J_l(\omega) = \boldsymbol{x}^T \boldsymbol{Z}_l(\omega_0) \left[\boldsymbol{Z}_l^T(\omega_0) \boldsymbol{Z}_l(\omega_0) \right]^{-1} \boldsymbol{Z}_l^T(\omega_0) \boldsymbol{x} \tag{10}$$

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3. The coefficient matrix has a block Toeplitz-plus-Hankel structure

$$\boldsymbol{Z}_l^T(\omega_0)\boldsymbol{Z}_l(\omega_0) = \begin{bmatrix} \boldsymbol{T}_l(\omega_0) & -\tilde{\boldsymbol{T}}_l(\omega_0) \\ \tilde{\boldsymbol{T}}_l(\omega_0) & \boldsymbol{T}_l(\omega_0) \end{bmatrix} + \begin{bmatrix} \boldsymbol{H}_l(\omega_0) & \tilde{\boldsymbol{H}}_l(\omega_0) \\ \tilde{\boldsymbol{H}}_l(\omega_0) & -\boldsymbol{H}_l(\omega_0) \end{bmatrix} \tag{11}$$

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4. The cost function $J_l(\omega)$ does not depend on the start index n_0.
5. If $n_0 = -(N-1)/2$, then $\tilde{\boldsymbol{T}}_l(\omega_0) = \tilde{\boldsymbol{H}}_l(\omega_0) = \boldsymbol{0}$, and

$$[\boldsymbol{T}_l(\omega_0) + \boldsymbol{H}_l(\omega_0)]\, \boldsymbol{a}_l(\omega_0) = \bar{\boldsymbol{w}}_l(\omega_0) \tag{12}$$

$$[\boldsymbol{T}_l(\omega_0) - \boldsymbol{H}_l(\omega_0)]\, \boldsymbol{b}_l(\omega_0) = -\tilde{\boldsymbol{w}}_l(\omega_0)\,. \tag{13}$$

where $\boldsymbol{Z}_l^T(\omega_0)\boldsymbol{x} = [\bar{\boldsymbol{w}}_l^T(\omega_0), \tilde{\boldsymbol{w}}_l^T(\omega_0)]^T$.

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8. Thus, solving the system for $l = L$ using a recursive Toeplitz-plus-Hankel solver gives the solutions for $l = 1, \ldots, L-1$ for free in the process.
9. Solving (14) therefore has a time complexity of $\mathcal{O}(l)$ when we have the solution to (14) for $l - 1$.

Fast ML/NLS

$$[\boldsymbol{T}_l(\omega_0) + \boldsymbol{H}_l(\omega_0)]\,\boldsymbol{a}_l(\omega_0) = \bar{\boldsymbol{w}}_l(\omega_0) \tag{14}$$

6. Fast algorithms for Toeplitz-plus-Hankel: Reduces time complexity from $\mathcal{O}(l^3)$ to $\mathcal{O}(l^2)$ (Merchant and Parks (1982), Gohberg and Koltracht (1989), Kailath and Sayed (1995)).
7. The solutions to all upper-left subsystems of (14) is a solution to a lower order.
8. Thus, solving the system for $l = L$ using a recursive Toeplitz-plus-Hankel solver gives the solutions for $l = 1, \ldots, L-1$ for free in the process.
9. Solving (14) therefore has a time complexity of $\mathcal{O}(l)$ when we have the solution to (14) for $l-1$.
10. The total time complexity is reduced to

$$\mathcal{O}(F \log F) + \mathcal{O}(FL) \tag{15}$$

Fast ML/NLS

Toeplitz-plus-Hankel Solver

$$[\boldsymbol{T}_l(\omega_0) + \boldsymbol{H}_l(\omega_0)]\,\boldsymbol{a}_l(\omega_0) = \bar{\boldsymbol{w}}_l(\omega_0) \tag{16}$$

- We use the recursive solver by Gohberg and Koltracht (1989).

Fast ML/NLS

Toeplitz-plus-Hankel Solver

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- Consists of a data independent and a data dependent step.

Fast ML/NLS
Toeplitz-plus-Hankel Solver

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- The data independent step consists in solving

$$[\boldsymbol{T}_l(\omega_0) + \boldsymbol{H}_l(\omega_0)]\,\gamma_l(\omega_0) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T . \tag{17}$$

for $\gamma_l(\omega_0)$ for $l = 1, \dots, L$.

Fast ML/NLS

Toeplitz-plus-Hankel Solver

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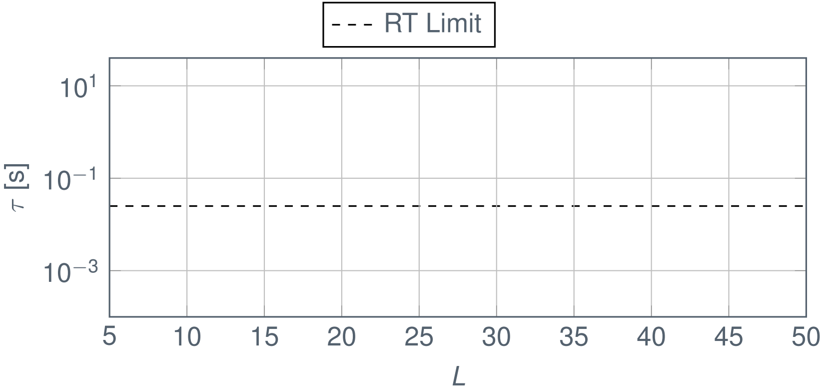
  for $\gamma_l(\omega_0)$ for $l = 1, \ldots, L$.
- The data independent step can be computed off-line. Requires memory to store $\gamma_l(\omega_0)$ for all frequencies and model orders.

Agenda

Computation Time vs. Model Order

MATLAB Implementation

Setup: $N = 200$ (25 ms @ $f_s = 8000$ Hz), $F = 5NL$, T420 laptop



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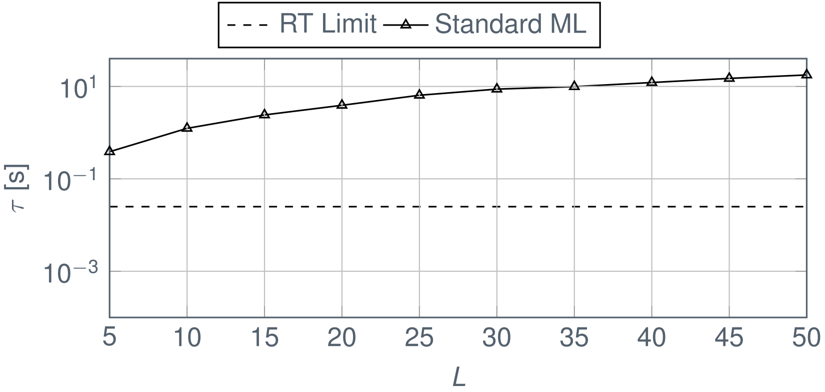

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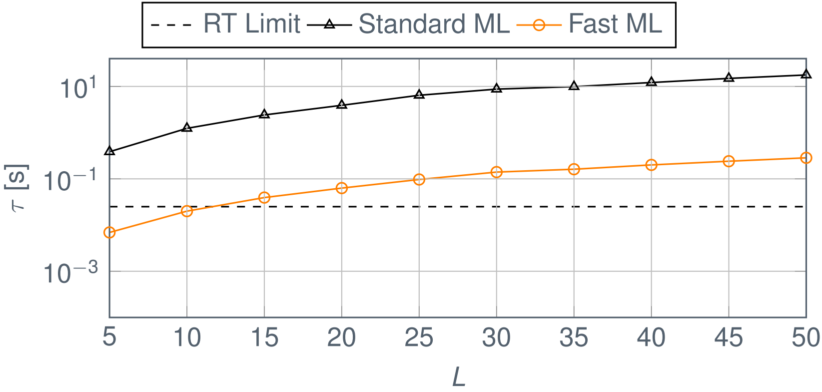

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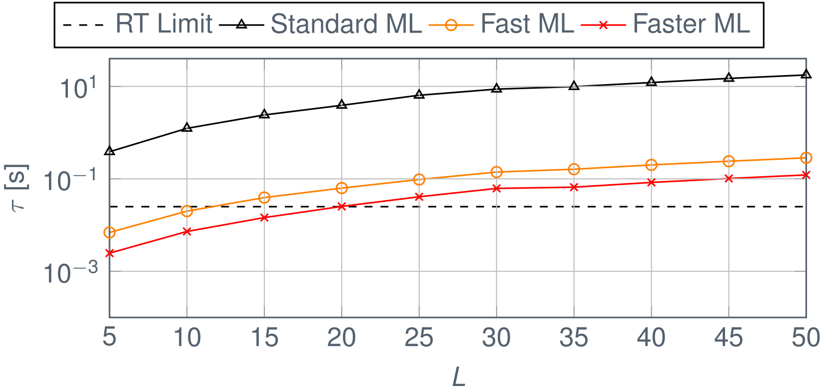

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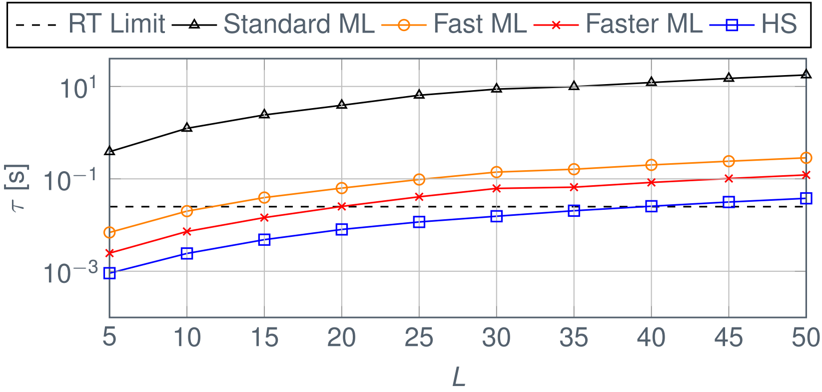

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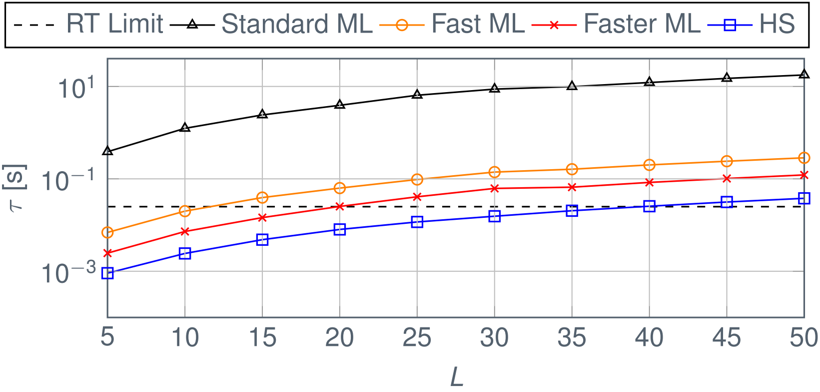
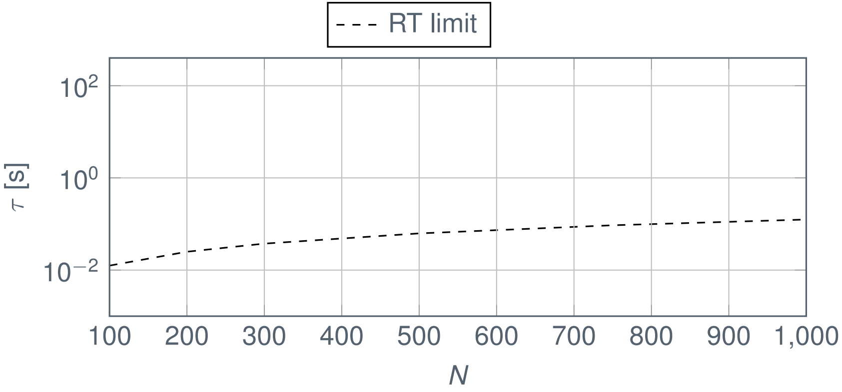


$\tau(\text{Standard ML}) \approx 60\tau(\text{Fast ML}) \approx 150\tau(\text{Faster ML}) \approx 500\tau(\text{HS})$

Computation Time vs. Data Size

MATLAB Implementation

Setup: $L = 30$, $F = 5NL$, T420 laptop



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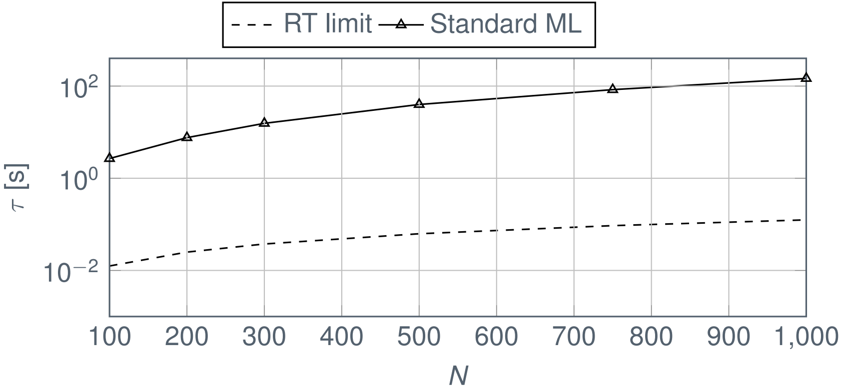

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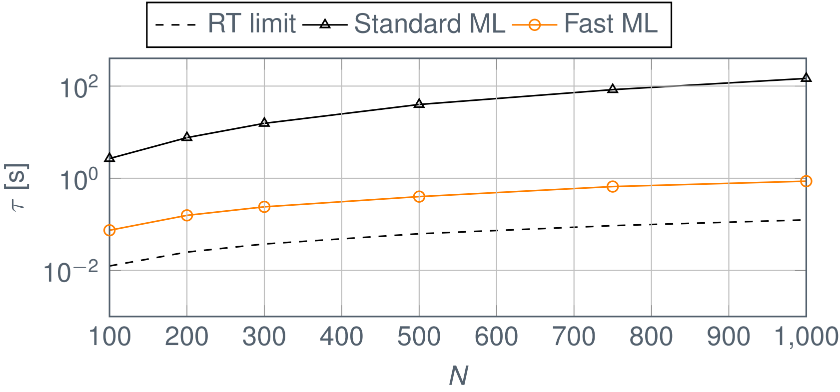

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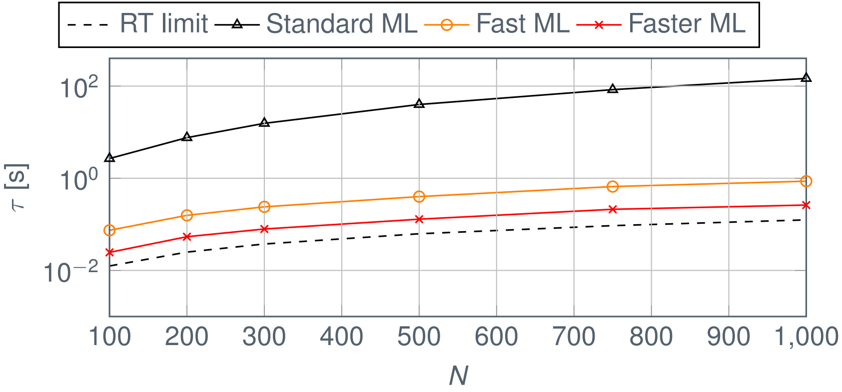

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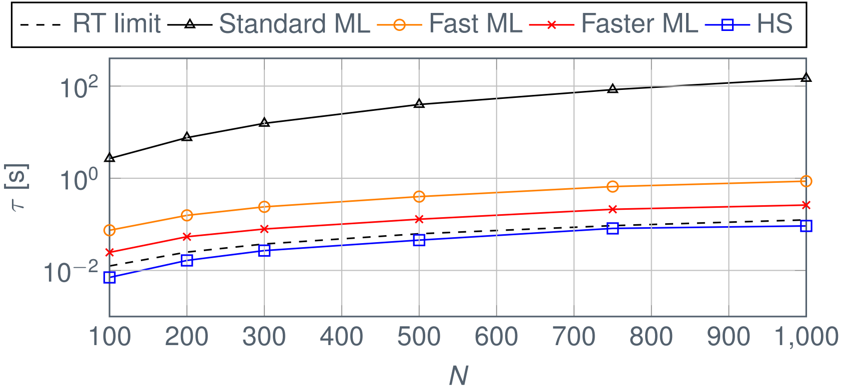
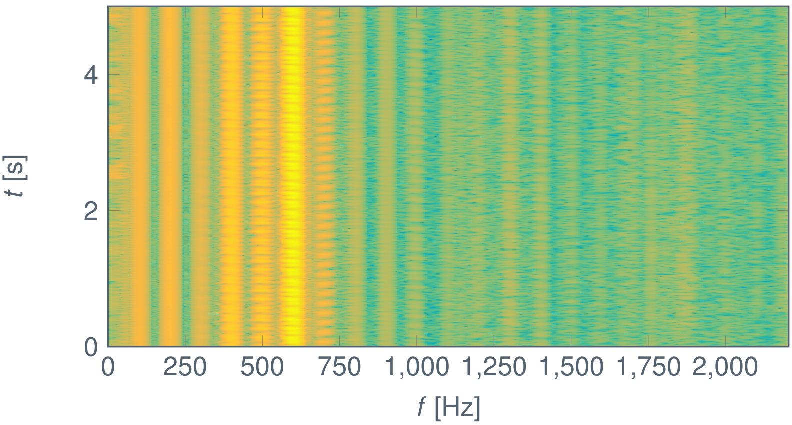


Washing Machine Example

Acoustic measurements by Brüel & Kjær

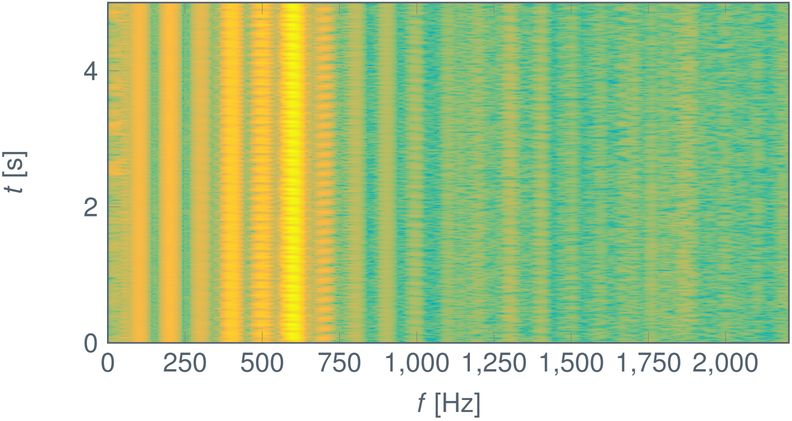
- $f_s = 44.1$ kHz



Washing Machine Example

Acoustic measurements by Brüel & Kjær

- $f_s = 44.1$ kHz
- $f_{rs} = 4410$ Hz, 60 ms windows, 15/16 overlap, and $L = 15$



Washing Machine Example

Acoustic measurements by Brüel & Kjær

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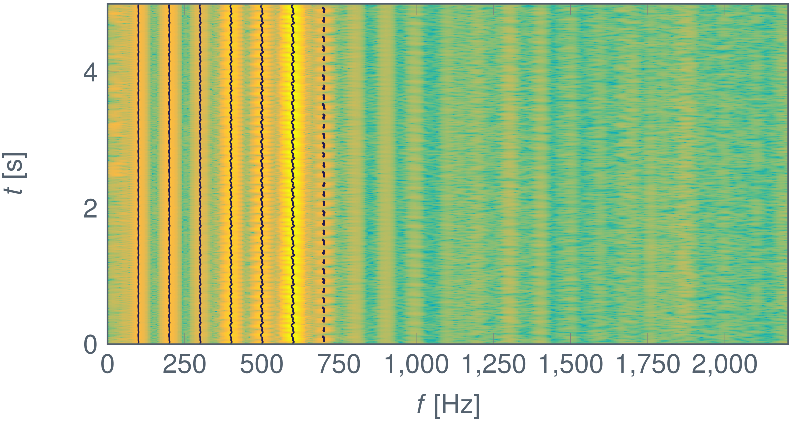
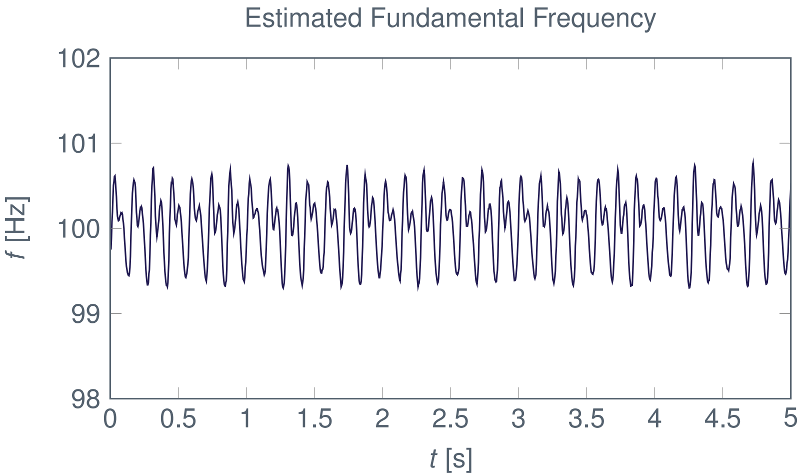
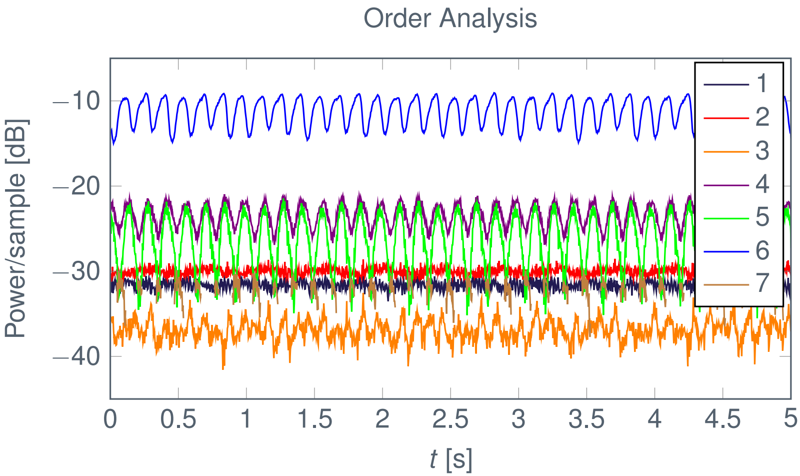


Washing Machine Example

Acoustic measurements by Brüel & Kjær



Washing Machine Example
Acoustic measurements by Brüel & Kjær



Agenda

Summary

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- We have proposed an algorithm that lower the complexity to

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  which is the same as that for harmonic summation.
- For a typical configuration, simulation studies show that the proposed algorithm is approximately 60-150 faster than the standard algorithm and $4 - 10$ times slower than harmonic summation.

Thanks for your attention!

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