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INDEPENDENT VERSUS REPEATED MEASUREMENTS: A PERFORMANCE QUANTIFICATION VIA STATE EVOLUTION

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1. Introduction and Contribution

1.1 Consider the following two situations (for illustration purposes):

I. Place cameras at different positions and take one shot of the object at each position.

II. Place one camera at one position and take several shots of the object.

The illumination might be stable or unstable.

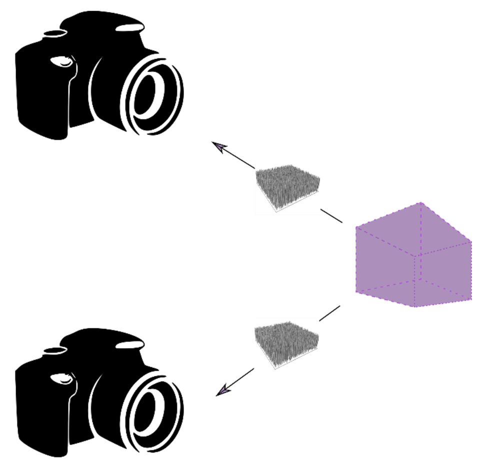

Fig 1a: Independent Measurements

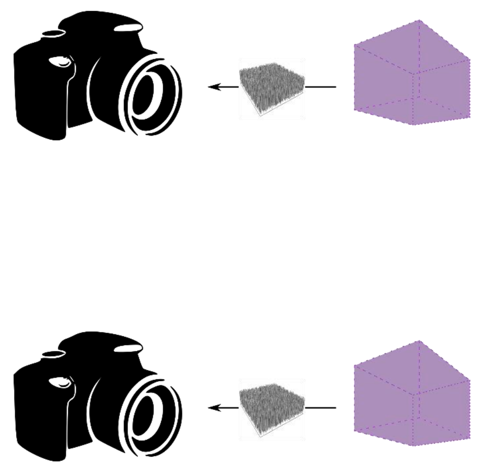

Fig 1b: Repeated Measurements

Generally speaking, the photo achieved from situation one should be better than the one achieved from situation two. But the exact understanding of the performance improvement is not available in the literature.

1.2 We identity AMP and state evolution can analyse the performances of above situations and exactly quantify the performance gap between them in the asymptotic regime.

2. Assumptions and Model Match

2.1 Both of the situations can be mathematically represented by the multiple measurement instances (MMI) model:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_K \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & & \\ & \ddots & \\ & & \boldsymbol{A}_K \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_K \end{bmatrix} + \begin{bmatrix} \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_K \end{bmatrix}$$

- ❑ $\boldsymbol{y}_k \in \mathbb{R}^m, \boldsymbol{A}_k \in \mathbb{R}^{m\times n}, \boldsymbol{x}_k \in \mathbb{R}^n, \forall k \in [K]$.
- ❑ $\boldsymbol{w}_k$'$s(\in \mathbb{R}^m)$ are additive white Gaussian noise vectors and mutually independent.

Addition assumption of the signal:

We assume common sparse supports model which means

$$\mathrm{supp}(\boldsymbol{x}_l) = \mathrm{supp}(\boldsymbol{x}_j), \forall l, j \in [K]$$

- ❑ $\mathrm{supp}(\boldsymbol{x}_l) = \{i: \quad x_{l,i} \neq 0, \forall i \in [n]\}$.
- ❑ $x_{l,i}$ is the i-th signal element form the l-th measurement instance.

and define the group single model

$$\boldsymbol{x}_{:,i} = \left[x_{1,i}, x_{2,i}, \dots, x_{K,i}\right]^T, \forall i \in [n]$$

- ❑ $\boldsymbol{x}_{:,i}$ contains the i-th components of all K measurement instances.

2.2 Model match for different situations

- For measurement matrices:
  1. Independent $\boldsymbol{A}_k$'s → distributed sensing (DS) model.
  2. Repeated $\boldsymbol{A}_k$'s → multiple measurement vectors (MMV) model.
- For signal vectors:
  1. Unstable illumination → correlated or independent amplitudes of nonzero coefficients of $\boldsymbol{x}_k$'s.
  2. Stable illumination → repeated $\boldsymbol{x}_k$'s.

3. AMP Algorithm and Its Extension

3.1 AMP algorithm has two good properties:

- ❑ Low computational complexity.
- ❑ Good performance guarantee (for Gaussian random matrices).

The original AMP algorithm was designed for the single measurement instance case ($K = 1$, e.g. generate the photo by one shot of the object), but the extension is easy to understand.

3.2 A heuristic model for the joint AMP (ignore noise)

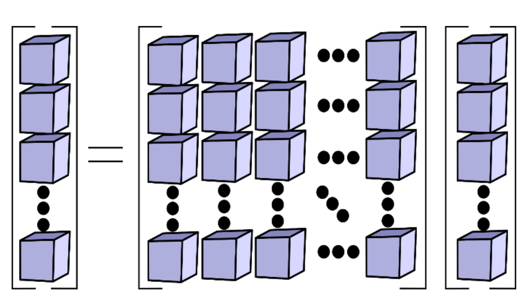

Fig 2a: Original Model

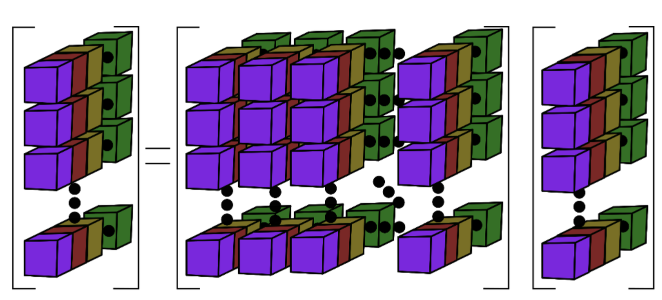

Fig 2b: Joint Model

3.3 AMP is an iterative algorithm:

$$\begin{aligned} \boldsymbol{x}^{t+1} &= \eta(\boldsymbol{A}^T r^t + \boldsymbol{x}^t), \\ r^t &= \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \frac{1}{\delta} r^{t-1} \langle \eta'(\boldsymbol{A}^T r^{t-1} + \boldsymbol{x}^{t-1}) \rangle. \end{aligned}$$

At each iteration, it updates the estimation and computes the variance of estimated errors. Both calculations work on the scalar variables ( ).

For joint operation, the algorithm will be similar but calculations work on the group signals ( ). By tracking the covariance matrix of the estimated errors, the performance of the system can be predicted.

4. Main Results

Assumptions:

- ❑ $\boldsymbol{A}_k$'s are independent (DS) or repeated (MMV) Gaussian random matrices.
- ❑ $\boldsymbol{x}_{:,i}$'s are independently and identically generated based on a specific distribution (e.g. Gaussian or Bernoulli-Gaussian) with mean zero and covariance matrix $\boldsymbol{\Sigma}_x$.
- ❑ $\boldsymbol{w}_k$'s are additive white Gaussian noise.

Outcomes:

- ❑ Asymptotic performance of DS and MMV models can be exactly quantified in the asymptotic regime:
  - ❖ Dimensions of the measurement matrices approach infinity proportionally.
  - ❖ The number of measurement instances remains a constant.
- ❑ DS outperforms MMV only when the signals are correlated.

5. Simulations

5.1 Theoretical analysis matches empirical results

- ❑ $K = 2$, $\boldsymbol{\Sigma}_x = \sigma_x^2[1, \rho; \rho, 1]$ and sparsity level is 0.2 for Bernoulli-Gaussian case.
- ❑ $n = 1000$ and the numerical results are obtained from the average of 100 trials.

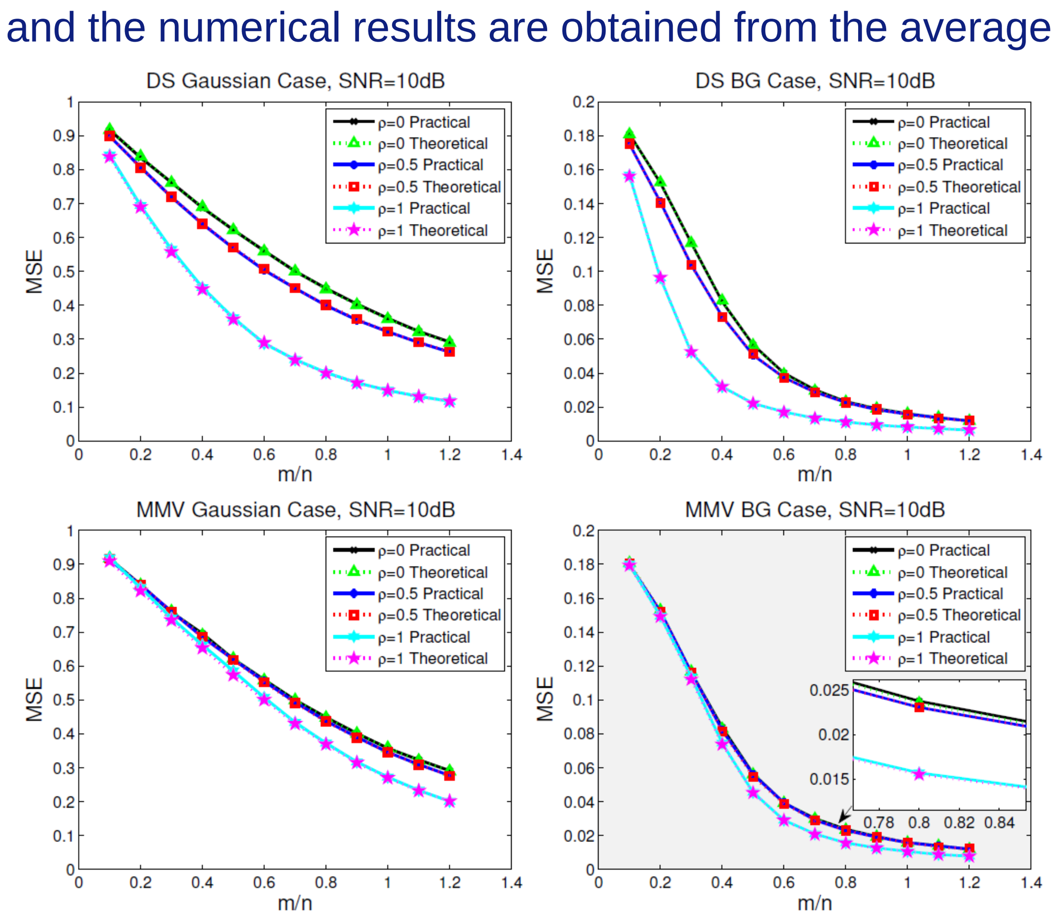

Fig 3: Simulation Results

5.2 Image example (106×114 pixels)

- ❑ $K = 2$ and SNR=10dB.
- ❑ Under-sampling rate $\delta = 0.7$.

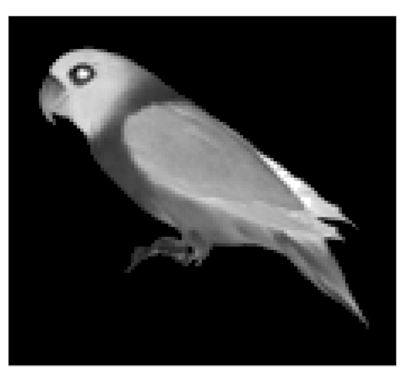

Fig 4a: Original

Fig 4b: Individual

Fig 4c: MMV

Fig 4d: DS

6. Key References

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