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Cramer-Rao Bound for Sparse Signals Fitting the Low-Rank Model with Small Number of Parameters

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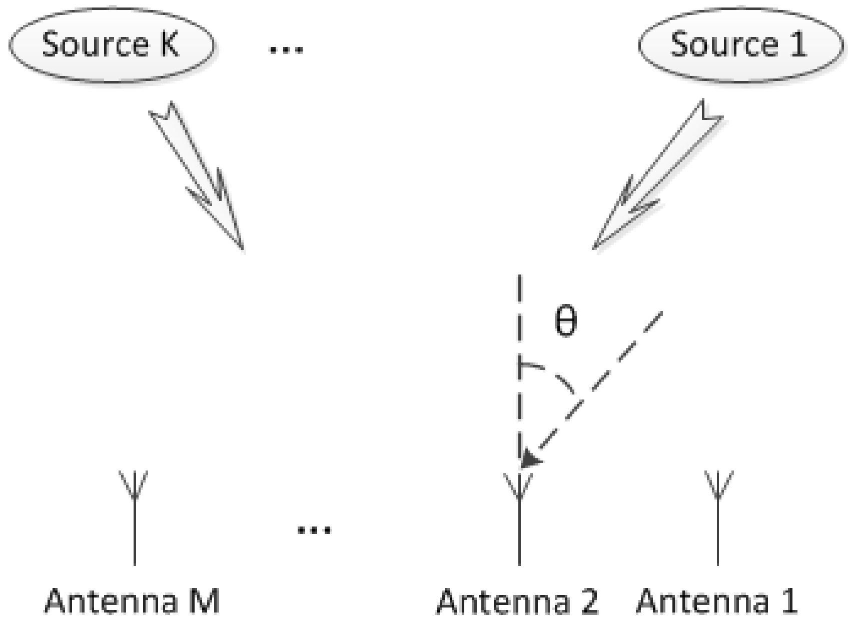
Summary

- Consider signals residing in a low-dimensional subspace characterized by a small number of parameters.
- Such signals with a sparse structure may be recovered from compressed measurements.
- The CRB gives a bound on the statistical performance of parameter estimation.
- The CRB can also be used to study the effect of compression and also to obtain the minimum required number of compressed samples.

Direction-of-Arrival (DOA) Estimation Application



$$\boldsymbol{a}(\theta) \triangleq \left[1,\ e^{-j2\pi(d/\lambda)\sin(\theta)}, \cdots,\ e^{-j2\pi(M-1)(d/\lambda)\sin(\theta)}\right]^T$$

$$\boldsymbol{A} = [\boldsymbol{a}(\theta_1),\ \boldsymbol{a}(\theta_2), \cdots,\ \boldsymbol{a}(\theta_K)]$$

Minimum Number of Compressed Samples

- If the number of compressed samples is less than or equal to the number of sources, the Fisher information matrix is singular.

- A singular FIM means that unbiased estimation of the entire parameter vector with finite variance is impossible.

- The converse does not hold in general.

- The minimum number of compressed samples for satisfactory performance depends on a specific performance criterion (such as probability of a subspace swap or the error of signal subspace estimation).

System Model

Low-rank system model

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{d}(t)$$

- $\boldsymbol{A}$ is a tall matrix.
- Measurement:

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{\Phi}\left(\boldsymbol{x}(t) + \boldsymbol{w}(t)\right) \\ &= \boldsymbol{\Phi}\boldsymbol{x}(t) + \boldsymbol{n}(t)\end{aligned}$$

- $\boldsymbol{\Phi}$ is the measurement matrix.
- No specific structure is assumed for matrix $\boldsymbol{\Phi}$.
- $\boldsymbol{\Phi}$ is treated as a deterministic matrix.
- $\boldsymbol{n}(t)$ is the additive noise with circularly-symmetric complex jointly Gaussian distribution

$$\mathcal{N}_C(\mathbf{0}, \boldsymbol{R})$$

$$\boldsymbol{R} = \sigma^2 \boldsymbol{\Phi}\boldsymbol{\Phi}^T$$

Derivation of the CRB

- Vector of parameters

$$\boldsymbol{\vartheta} \triangleq \left[\bar{\boldsymbol{d}}^T(1), \tilde{\boldsymbol{d}}^T(1), \cdots, \bar{\boldsymbol{d}}^T(N), \tilde{\boldsymbol{d}}^T(N), \boldsymbol{\Omega}^T\right]^T$$

- $\bar{\boldsymbol{d}}(t)$ and $\tilde{\boldsymbol{d}}(t)$ the real and imaginary parts of $\boldsymbol{d}(t)$.
- $\boldsymbol{\Omega} \triangleq [\omega_1, \cdots, \omega_P]^T$ contains the unknown parameters of matrix $\boldsymbol{A}$.
- The CRB is given by

$$\mathrm{CRB}(\boldsymbol{\vartheta}) = \boldsymbol{I}^{-1}(\boldsymbol{\vartheta})$$

- The Fisher information matrix is given by

$$\boldsymbol{I}(\boldsymbol{\vartheta}) = E\left\{\psi\psi^T\right\}$$

$$\psi \triangleq \partial LL / \partial \boldsymbol{\vartheta}$$

$$LL \triangleq \ln p\left(\boldsymbol{y}(1), \cdots, \boldsymbol{y}(N) \,|\, \boldsymbol{\vartheta}\right)$$

Numerical Example

- DOA estimation of 11 equally spaced sources form 20 to 50 degrees.

- Uniform linear array with 50 antenna elements.

- Steering vector

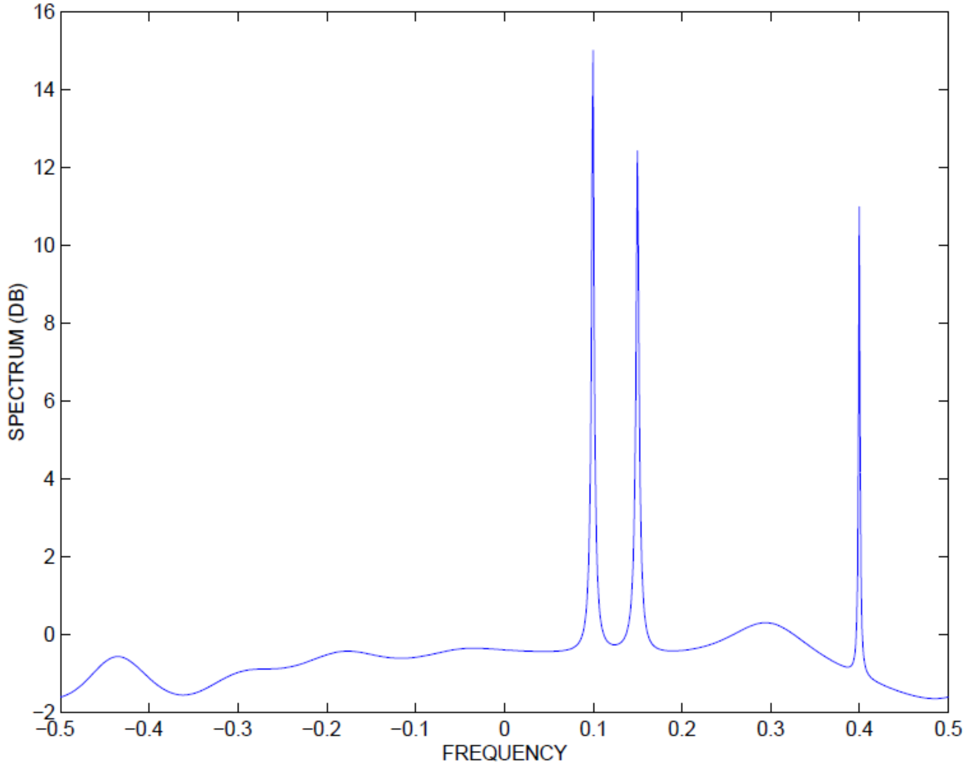
$$\boldsymbol{a}(\omega) \triangleq \left[1, e^{-j2\pi(d/\lambda)\sin(\omega)}, \cdots, e^{-j2\pi(N_x-1)(d/\lambda)\sin(\omega)}\right]^T$$

- *N = 10* snapshots.

- The source vector $\boldsymbol{d}(t)$ is distributed according to

$$\mathcal{N}_C(\mathbf{0}, \sigma_s^2 \boldsymbol{I}_K)$$

Spectral Estimation Application



$$x(n) = \sum_{k=1}^{K} d_k e^{-j\omega_k n} \quad \Longrightarrow \quad \boldsymbol{x} = \boldsymbol{A}\boldsymbol{d}$$

CRB for the parameters of matrix A

$$\mathrm{CRB}^{-1}(\boldsymbol{\Omega}) = 2\sum_{t=1}^{N} Re\Big\{\boldsymbol{D}^H(t)\boldsymbol{\Phi}^T\boldsymbol{R}^{-1}\Big(\boldsymbol{I}_{N_y} - \boldsymbol{B}\left(\boldsymbol{B}^H\boldsymbol{R}^{-1}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^H\boldsymbol{R}^{-1}\Big)\boldsymbol{\Phi}\boldsymbol{D}(t)\Big\}$$

- where

$$\boldsymbol{D}(t) \triangleq \left[\frac{\partial \boldsymbol{A}}{\partial \omega_1}\boldsymbol{d}(t), \cdots, \frac{\partial \boldsymbol{A}}{\partial \omega_P}\boldsymbol{d}(t)\right] = \left[\frac{\partial \boldsymbol{A}}{\partial \omega_1}, \cdots, \frac{\partial \boldsymbol{A}}{\partial \omega_P}\right](\boldsymbol{I}_P \otimes \boldsymbol{d}(t))$$

$$\boldsymbol{B} \triangleq \boldsymbol{\Phi}\boldsymbol{A}$$

CRB for estimating the source at 35 degrees