Sampling solutions of Schrödinger equations on combinatorial graphs

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Introduction

During the last years applications to information theory, signal analysis, image processing, computer sciences, learning theory, astronomy triggered new developments of harmonic analysis on combinatorial graphs.

In particular, a sampling theory of functions on combinatorial graphs recently became a rather active field of research.

In my talk I consider sampling of stationary and *non-stationary* signals on graphs.

By non-stationary signals we understand signals which depend on time and whose evolution in time is governed by a Schrödinger type equation with a combinatorial Laplace operator on the right-hand side.

It will be shown that solutions of such equations with bandlimited initial data can be perfectly reconstructed from their samples on the graph and on the time axis.

Introduction

We consider a finite graph $G = (V, E)$, where $V = V(G)$ is the set of $|V|$ vertices or nodes and $E = E(G)$ is the set of edges or links connecting these vertices. The weight of the edge connecting two nodes u and v is denoted by $w(u, v)$. The degree $\mu(v)$ of the vertex v is the sum of the edge weights incident to node v.

The adjacency matrix W of the graph is an $|V| \times |V|$ matrix such that $W(u, v) = w(u, v)$.
The Hilbert space $L_2(G)$ is the set of all complex valued functions f on $V(G)$

$$f : V \to \mathbb{C},$$

with the following inner product

$$\langle f, g \rangle_{L_2(G)} = \langle f, g \rangle = \sum_{v \in V(G)} f(v) \overline{g(v)} \, \mu(v). \tag{1}$$

For such graph the weighted Laplace operator Δ is introduced via

$$(\Delta f)(v) = \sum_{u \in V(G)} (f(v) - f(u)) w(v, u) \ . \tag{2}$$

The Laplacian on a finite graph is a positive-semidefinite self-adjoint bounded operator.

Bandlimited functions on graphs

In what follows I will develop Shannon-type sampling of functions bandlimited on graphs.

Let $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{|V|-1}$ be the set of eigenvalues of Δ and let $e_{\lambda_0}, ..., e_{\lambda_{|V|-1}}$ be an orthonormal complete set of eigenfunctions. For a function $f \in L_2(G)$ it's Fourier coefficients $c_j(f)$ are defined as usual

$$c_j(f) = \sum_{v \in V(G)} f(v)\overline{e_{\lambda_j}(v)}.$$

Bandlimited functions on graphs

Definition

A function (signal) f on a finite weighted graph G is said to be ω-bandlimited if it has expansion

$$f = \sum_{\lambda_j \leq \omega} c_j e_{\lambda_j}.$$

The space of ω-bandlimited signals is also called the **Paley-Wiener space** and is denoted by $\mathbf{E}_\omega(G)$.

Sampling of stationary signals bandlimited on graphs

At this point I am going to discuss stationary signals on graphs.

It will be shown that analysis of lower frequencies on a graph can be performed on a smaller subgraph. Note that in many situations lower frequencies are more informative while higher frequencies are usually associated with noise.

Sampling(uniqueness) sets

Definition

A subset of vertices $U \subset V(G)$ is a sampling set (uniqueness set) for a space $\mathbf{E}_\omega(G)$, $\omega > 0$, if for any two signals from $\mathbf{E}_\omega(G)$, the fact that they coincide on U implies that they coincide on $V(G)$.

Frame inequalities

The next important theorem shows that every sampling set is associated with a family of frame inequalities.

Theorem

A set $U \subset V(G)$ is a sampling set for a space $\mathbf{E}_\omega(G)$ if and only if there exists a constant $B = B(U)$ such that the following frame inequalities hold

$$\|f\|^2 \leq \sum_{w \in S} |f(w)|^2 \leq B\|f\|^2$$

for all functions in $\mathbf{E}_\omega(G)$.

Reconstruction using frames

These frame inequalities imply that projections of delta functions $\delta_w, \;\; w \in U$, on $\mathbf{E}_\omega(G)$ form a frame in $\mathbf{E}_\omega(G)$.
The next Corollary follows from the general theory of Hilbert frames.

Corollary

If U is a sampling set for the space $\mathbf{E}_\omega(G)$ then there exists a frame $\{\Phi^\omega_u\}_{u\in U}$ in the space $\mathbf{E}_\omega(G)$ such that for any $f \in \mathbf{E}_\omega(G)$ the following reconstruction formula holds

$$f(v) = \sum_{u\in U} f(u)\Phi^\omega_u(v), \quad v \in V(G). \tag{3}$$

The Cauchy problem for Schrd̈inger equation on graphs

We consider the following Cauchy problem for the Schrödinger type equation

$$\frac{dg(t,v)}{dt} = i\Delta g(t,v), \quad g(0,v) = f(v), \tag{4}$$

where $v \in V(G), t \in \mathbf{R}$, $i^2 = -1$.
If the initial function f belongs to the domain of the operator Δ then the unique solution to this problem is given by the formula $g(v,\ t) = e^{it\Delta} f(v),\ -\infty < t < \infty,\ \ v \in V(G)$, where $e^{it\Delta}$ is a group of unitary operators in $L_2(G)$.

Sampling solutions of Schrödinger equation

We have the following result in which equally spaced sampling points are considered.

Theorem

Assume that $f \in PW_\omega(G)$ *and* $S \subset V(G)$ *is a sampling set for* $PW_\omega(G)$. *Then the solution* $g(t, v) = e^{it\Delta} f(v)$ *to the Cauchy problem (9) at any point* $(t, v) \in \mathbf{R} \times V(G)$ *is completely determined by the set of samples* $g(k\pi/\sigma, s),\ k \in \mathbf{Z},\ s \in S$ *for any* $\sigma > \omega$.

Sampling solutions of Schrödinger equation

Theorem

Moreover, the explicit reconstruction formula is given for any vertex $s \in S \subset V(G)$ *by the formula for all* $t \in \mathbf{R}$

$$g(t,s) = \sum_{k \in \mathbf{Z}} g\left(\frac{k\pi}{\sigma}, s\right) sinc\, \frac{\sigma}{\pi}\left(t - \frac{k\pi}{\sigma}\right), \quad (5)$$

were convergence of the series is uniform convergence on compact subsets of $\mathbf{R}$

Sampling solutions of Schrödinger equation

To extend function (5) to entire graph one can use for every $t \in \mathbf{R}$ the formula

$$g(t, v) = \sum_{s \in S} g(t, s)\Phi_s^{\omega}(v),$$

where $\{\Phi_s^{\omega}\}_{s \in S}$ is a frame in $PW_{\omega}(G)$ described in (3). If the graph is finite the sum in the last formula is finite. If the graph is infinite then the converge holds in the norm of $L_2(G)$.

The next Corollary follows from the fact that if Δ is a bounded operator then every function in $L_2(G)$ belongs to $PW_\omega(G)$ for any $\omega \geq \|\Delta\|$.

Corollary

If operator Δ is bounded in the space $L_2(G)$ then the previous Theorem holds for every $f \in L_2(G)$ as long as $\sigma > \omega \geq \|\Delta\|$.

Valiron-Tschakaloff sampling/interpolation formula

The next theorem is a generalization of what is known as the Valiron-Tschakaloff sampling/interpolation formula.

Let us remind that sinc(t) is defined as $\frac{\sin \pi t}{\pi t}$, if $t \neq 0$, and 1, if $t = 0$.

Valiron-Tschakaloff sampling/interpolation formula

Theorem

For $f \in \mathbf{E}_{\omega}(G), \quad \omega > 0$, *we have for all* $t \in \mathbf{R}$

$$g(t,\ v) = it\ sinc\left(\frac{\omega t}{\pi}\right)\Delta f(v) + sinc\left(\frac{\omega t}{\pi}\right) f(v) +$$

$$\sum_{k\in\mathbf{Z},\ k\neq 0} \frac{\omega t}{k\pi} sinc\left(\frac{\omega t}{\pi} - k\right) g\left(\frac{k\pi}{\omega},\ v\right), \qquad (6)$$

where $g(t, v) = e^{it\Delta} f(v)$ *and convergence is in the space of abstract functions* $L_2\left((-\infty, \infty),\ L_2(G)\right)$ *with the regular Lebesgue measure.*

Again, we consider the following Cauchy problem

$$\frac{dg(t,v)}{dt} = i\Delta g(t,v), \quad g(0,v) = f(v) \in L_2(G), \tag{7}$$

where $v \in V(G), t \in \mathbf{R}$.
The unique solution to this problem is given by the formula $g(t,\ v) = e^{it\Delta} f(v),\ -\infty < t < \infty,\ v \in V(G)$, where $e^{it\Delta}$ is a group of unitary operators in $L_2(G)$.

We assume that for a $T > 0$ the solution

$$g(T,\ v) = e^{iT\Delta} f(v), \quad v \in S \subset V(G),$$

is known on a subset of vertices S.

The goal of this section is to describe an algorithm which allows for approximate reconstruction of the initial data

$$f(\cdot) = g(0, \cdot)$$

using a single sample $g(T, \cdot) = e^{it\Delta} f(\cdot),\ T \neq 0$.

Let δ_v be a Dirac measure concentrated at a vertex $v \in V(G)$.

Lemma

Let $f \in PW_\omega(G)$ and S be a sampling set for $PW_\omega(G)$. Let

$$\mathcal{P}^\omega : L_2(G) \to PW_\omega(G)$$

be orthogonal projector. Then $\{\mathcal{P}^\omega \delta_s\}_{s \in S}$ is a Parseval frame in $PW_\omega(G)$. Which means that the following formula holds for any $f \in PW_\omega(G)$

$$f = \sum_{s \in S} \langle f, \mathcal{P}^\omega \delta_s \rangle \mathcal{P}^\omega \delta_s \tag{8}$$

Since for any $t \in \mathbf{R}$ one has

$$\left\langle e^{it\Delta} f, \delta_v \right\rangle = \left\langle f, e^{-it\Delta} \delta_v \right\rangle,$$

and since $PW_\omega(G)$ is invariant with respect to $e^{it\Delta}$ one has the following:

For any $T > 0$ the set

$$\left\{ e^{-iT\Delta} \mathcal{P}^\omega \delta_s \right\}_{s \in S}$$

is a Parseval frame in $PW_\omega(G)$.

Note that

$$f = \sum_{s \in S} \left\langle f, e^{-iT\Delta} \mathcal{P}^{\omega} \delta_s \right\rangle e^{-iT\Delta} \mathcal{P}^{\omega} \delta_s =$$

$$\sum_{s \in S} \left\langle e^{iT\Delta} f, \delta_s \right\rangle e^{-iT\Delta} \mathcal{P}^{\omega} \delta_s$$

Thus if $g(t, v)$, $v \in V(G), t \in \mathbf{R}$ is the solution of the Cauchy problem

$$\frac{dg(t, v)}{dt} = i\Delta g(t, v), \quad g(0, v) = f(v) \in L_2(G), \tag{9}$$

then we can formulate the following result.

Theorem

If $f \in PW_{\omega}(G)$ and S is a sampling set for $PW_{\omega}(G)$ then for any $T > 0$ the initial condition f can be reconstructed from the values of $g(T, s)$ on the set S:

$$f(v) = \sum_{s \in S} g(T, s) \left(e^{-iT\Delta} \mathcal{P}^{\omega} \delta_s \right)(v), \quad v \in V(G).$$

In order to avoid a costly and difficult procedure of computing the operator $e^{-iT\Delta}$ we are going to show that for any $\epsilon > 0$ there exists such functions $\{\psi_s^\omega\}_{s \in S}$ for which

$$\|\theta_s^\omega - \psi_s^\omega\| = \|e^{-iT\Delta}\mathcal{P}^\omega \delta_s - \psi_s^\omega\| < \epsilon, \tag{10}$$

for all $s \in S$. For a given $N \in \mathbf{N}$ consider the polynomial

$$\mathcal{T}_N(x) = \sum_{n=0}^{N} \frac{(-i)^n (x - \frac{\alpha}{2})^n}{n!}.$$

One can prove the next statement.

Lemma

For a given $\alpha > 0$ and $\epsilon > 0$, if $N \geq \max\left\{\alpha e^2/2,\ \ln(\epsilon^{-1})\right\}$ then

$$\sup_{x \in [0,\alpha]} \left| e^{-ix} - \mathcal{T}_N(x) \right| < \epsilon$$

Using this lemma and the spectral theorem we obtain that for all $t \in [0, \alpha]$ the following estimate holds.

$$\|e^{-it\Delta} - \mathcal{T}_N(\Delta)\| < \epsilon.$$

Now, in order to satisfy (10) we introduce

$$\psi_s^{\omega} = \mathcal{T}_N(\Delta)\mathcal{P}^{\omega}\delta_s.$$

Clearly, for a sufficiently small ϵ the set $\{\psi_s^\omega\}_{s\in S}$ will be a frame in $PW_\omega(G)$ which means

$$f = \sum_{s\in S} \langle f, \psi_s^\omega \rangle \, \Psi_s^\omega,$$

where $\{\Psi_s^\omega\}_{s\in S}$ is a frame dual to $\{\psi_s^\omega\}_{s\in S}$. Since for every $f \in PW_\omega(G)$

$$\left\langle e^{iT\Delta} f, \delta_s \right\rangle = \left\langle f, e^{-iT\Delta} \mathcal{P}^\omega \delta_s \right\rangle \approx \langle f, \psi_s^\omega \rangle$$

we obtain that for

$$\widetilde{f} = \sum_{s \in S} \left\langle e^{iT\Delta} f, \delta_s \right\rangle \Psi_s^{\omega} = \sum_{s \in S} \left\langle f, e^{-iT\Delta} \delta_s \right\rangle \Psi_s^{\omega}$$

the following approximate formula holds

$$f(v) \approx \sum_{s \in S} \left\langle e^{iT\Delta} f, \delta_s \right\rangle \Psi_s^{\omega}(v) = \widetilde{f}(v), \ \ v \in V(G). \tag{11}$$

After all the error of approximation (11) can be estimated as follows

$$\|f-\widetilde{f}\| = \left\| \sum_{s\in S} \left\langle f, e^{-iT\Delta}\delta_s - \psi_s^{\omega} \right\rangle \Psi_s^{\omega} \right\| \leq$$

$$\epsilon\|f\| \sum_{s\in S} \|\Psi_s^{\omega}\| \leq C|V(G)|\epsilon\|f\|. \tag{12}$$

Theorem

Assume that graph G is finite. If $f \in PW_\omega(G)$ and for a $T \neq 0$ the function $g(T, \cdot) = e^{iT\Delta} f(\cdot)$ is known on a set S which is a sampling set for $PW_\omega(G)$ then f can be approximately reconstructed by the formula (11). Moreover, there exists a constant $C > 0$ such that for any $\epsilon > 0$ the error of approximation is given by (12).

Conclusion

In the present paper we consider non-stationary signals which propagate on a combinatorial graph and whose evolution is governed by a Schrödinger type equation with a combinatorial Laplace operator on the right side. It is shown that such signals can be perfectly reconstructed from their samples on the graph and on the time axis.
Another new result shows how to obtain an approximate reconstruct of the initial function of a solution of the Cauchy problem using just one signal sample of this solution.

I. Pesenson, *Sampling in Paley-Wiener spaces on combinatorial graphs,* Trans. Amer. Math. Soc. **360** (2008), 5603-5627.
I. Pesenson, M. Pesenson , *Sampling, filtering and sparse approximations on graphs*, JFAA, (2010), Vol.16, no. 6, 921-942.
Führ, Hartmut; Pesenson, Isaac Z., *Poincare and Plancherel-Polya inequalities in harmonic analysis on weighted combinatorial graphs*, SIAM J. Discrete Math. 27 (2013), no. 4, 2007-2028.
I. Pesenson, *Sampling formulas for one-parameter groups of operators in Banach spaces*, Sampl. Theory Signal Image Process. Vol. 14 (2015), no. 1, 1-16.