Atypicality for Vector Gaussian Models

Elyas Sabeti, Anders Høst-Madsen

- Dept. of EE, University of Hawaii

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  - Medical sensors
  - Genetics
  - Surveillance: NSA
  - Environmental sensors

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Applications

- Medical
  - Most sensor data is indicative of normal
  - The rare event is indicative of decease
- Other
  - Gambling fraud or malfunction
  - Credit card fraud
  - Accounting, IRS
  - Computer network intrusion
  - Environmental monitoring
  - Electric power grids
  - Plant monitoring

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Anomaly Detection with Universal Source Coding

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- Aim
  - Theoretically well-founded approach to anomaly detection with information theory

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  - Entropy $H(X) \rightarrow$ Shortest codelength
  - Mutual Information $I(X;Y) \rightarrow$ Channel capacity

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A Mathematical Theory of Communication

By C. E. SHANNON

Introduction

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

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- This work is based on an assumption that information is fundamental
  - Information measure is not <u>a</u> measure but <u>the</u> measure

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- Random Sequence

$$\exists c > 0 \forall n > 1 : K(x[1], \ldots, x[n]) \geq n - c$$

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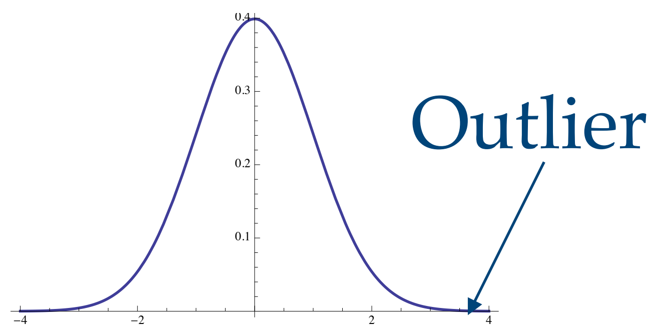
  - Low likelihood, rarity: $C_t(x)$ large



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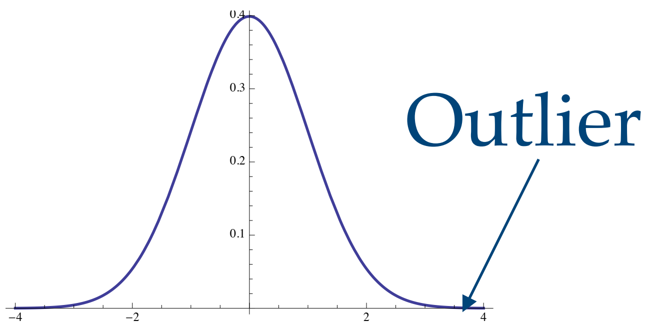


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10011011010100001
11111111111111111
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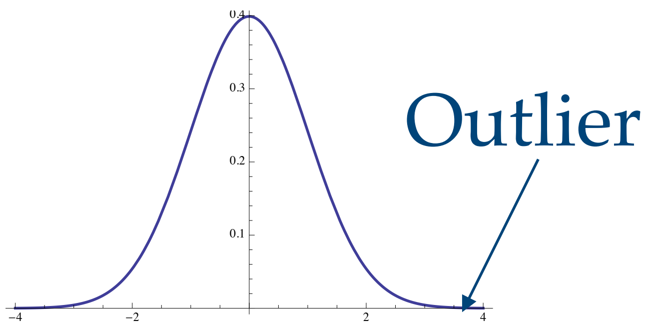

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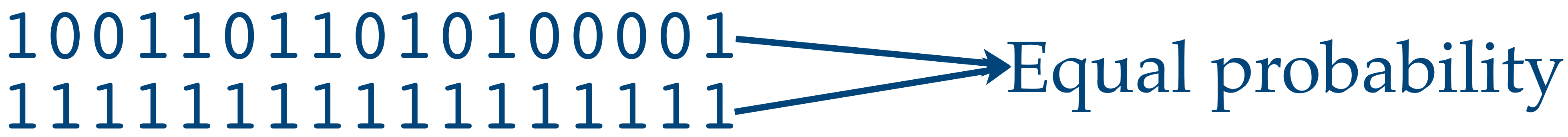

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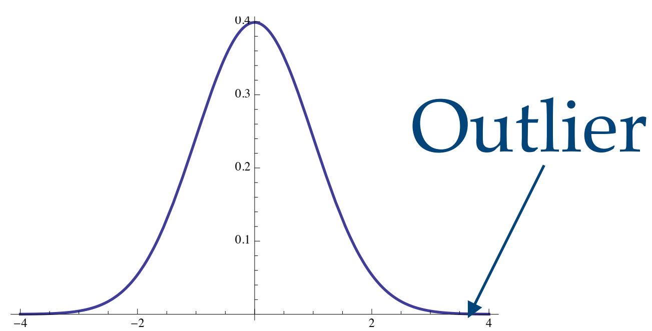

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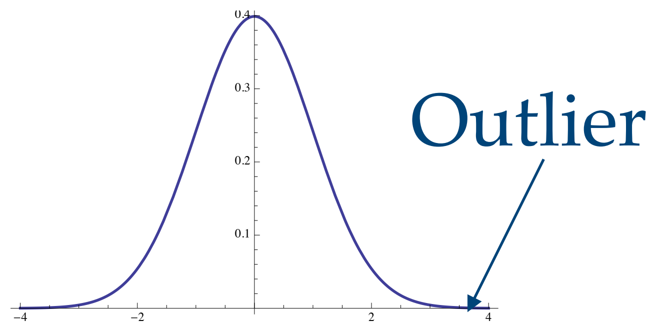

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→ Equal probability

  - $C_t(x)$ same, but $C_a(x)$ different
- Also prioritizes these cases
  - The larger $C_t(x) - C_a(x)$ the more atypical

Binary IID sequences

1000011010011111111111100001010101111110001

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- Total codelength

$$L_{\hat{p}}(l) = lH(\hat{p}) + \frac{3}{2}\log l + \tau$$

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  - Codelength: $L_{\hat{p}}(l) = lH(\hat{p}) + \dfrac{3}{2}\log l + \tau$
- Atypicality criterion

$$D(\hat{p}\|p) > \frac{\tau + \frac{3}{2}\log l}{l}$$

Theoretical Analysis

- The probability P_A that a sequence of length l is classified as atypical is bounded by

$$P_A \leq 2^{-\tau+1}\frac{1}{l^{3/2}}K(l,\tau), \quad \forall\tau : \lim_{l\to\infty} K(l,\tau) = 1$$

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- Consider the case $p=\frac{1}{2}$. The probability $P_A(X_n)$ that a given sample X_n is part of an atypical subsequence of any length is upper bounded by

$$P_A(X_n) \leq (K_1\sqrt{\tau} + K_2)2^{-\tau}$$

for some constants K_1, K_2

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  - Lossless audio coding (MPEG-4 ALS, Apple Lossless)
- Abstract encoding
  - Fixed point, r bits after ., unlimited bits prior
  - Codelength (Rissanen)

$$L(x) = -\log \int_x^{x+2^{-r}} f(t)dt \approx -\log(f(x)) + r$$

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- Parametric model $f(\mathbf{x}|\boldsymbol{\theta})$
  - Need to encode data and parameters
  - Rissanen's MDL: $L = -\log f(\mathbf{x}|\hat{\boldsymbol{\theta}}_{\text{ML}}) + \frac{k}{2}\log l$

Vector Gaussian case

- Model
$$\mathbf{x}[n] = \mathbf{s}(\boldsymbol{\theta}) + \mathbf{w}[n]$$
where $\mathbf{w}[n] \sim \mathcal{N}(0, \boldsymbol{\Sigma}), \mathbf{s}(\boldsymbol{\theta}) k$-parameter
- Used to find atypical *relationships* between data streams
- **Theorem**: Probability of intrinsically atypical sequence
$$\limsup_{l\to\infty} \frac{\ln P_A(l)}{\frac{k+2}{2}\ln l} \leq 1$$
- Or
$$P_A(l) \lesssim l^{\frac{k+2}{2}}$$

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Proof

- Atypicality criterion

$$r(\mathbf{x}) = -\log \frac{f(\mathbf{x}|\hat{\boldsymbol{\theta}})}{f(\mathbf{x}|\boldsymbol{\theta})} \geq \tau + \frac{k+2}{2}\log l$$

- Chernoff bound

$$P\left(r(\mathbf{x}) \geq \tau + \frac{k+2}{2}\log l\right) \leq \exp(-s(\tau + \frac{k+2}{2}\log l))M_r(s)$$

- Need to prove $M_r(s) = E[e^{sr}] \leq K < \infty$ independent of l for $s < \ln 2$

Proof

- Need to prove $M_r(s) = E[e^{sr}] \le K < \infty$ independent of l for $s < \ln 2$

$$
\begin{aligned}
-\ln \frac{p(\mathbf{x}|\hat{\boldsymbol{\theta}})}{p(\mathbf{x}|\boldsymbol{\theta})} &= \frac{1}{2}\sum_{n=1}^{l} \mathbf{x}[n]^T \boldsymbol{\Sigma}^{-1} \mathbf{x}[n] \\
&\quad - \frac{1}{2}\sum_{n=1}^{l} \left(\mathbf{x}[n] - \mathbf{s}(\hat{\boldsymbol{\theta}})\right)^T \boldsymbol{\Sigma}^{-1} \left(\mathbf{x}[n] - \mathbf{s}(\hat{\boldsymbol{\theta}})\right) \\
&\le \frac{1}{2l}\left(\sum_{n=1}^{l} \mathbf{x}[n]\right)^T \boldsymbol{\Sigma}^{-1} \left(\sum_{n=1}^{l} \mathbf{x}[n]\right)
\end{aligned}
$$

- Here $t = \sum_{n=1}^{l} \mathbf{x}[n]$ is sufficient statistic

Proof

- Need to prove $M_r(s) = E[e^{sr}] \leq K < \infty$ independent of l for $s < \ln 2$

$$E[e^{sr}] \leq \frac{1}{(2\pi)^{l/2}\sqrt{l\det\mathbf{\Sigma}}} \int \exp\left(\frac{s}{2l\ln 2}\mathbf{t}^T\mathbf{\Sigma}^{-1}\mathbf{t}\right)$$
$$\times \exp\left(-\frac{1}{2l}\mathbf{t}^T\mathbf{\Sigma}^{-1}\mathbf{t}\right) d\mathbf{t}$$
$$\leq K$$

- Here $t = \sum_{n=1}^{l} \mathbf{x}[n]$ is sufficient statistic

Example: S & P 500

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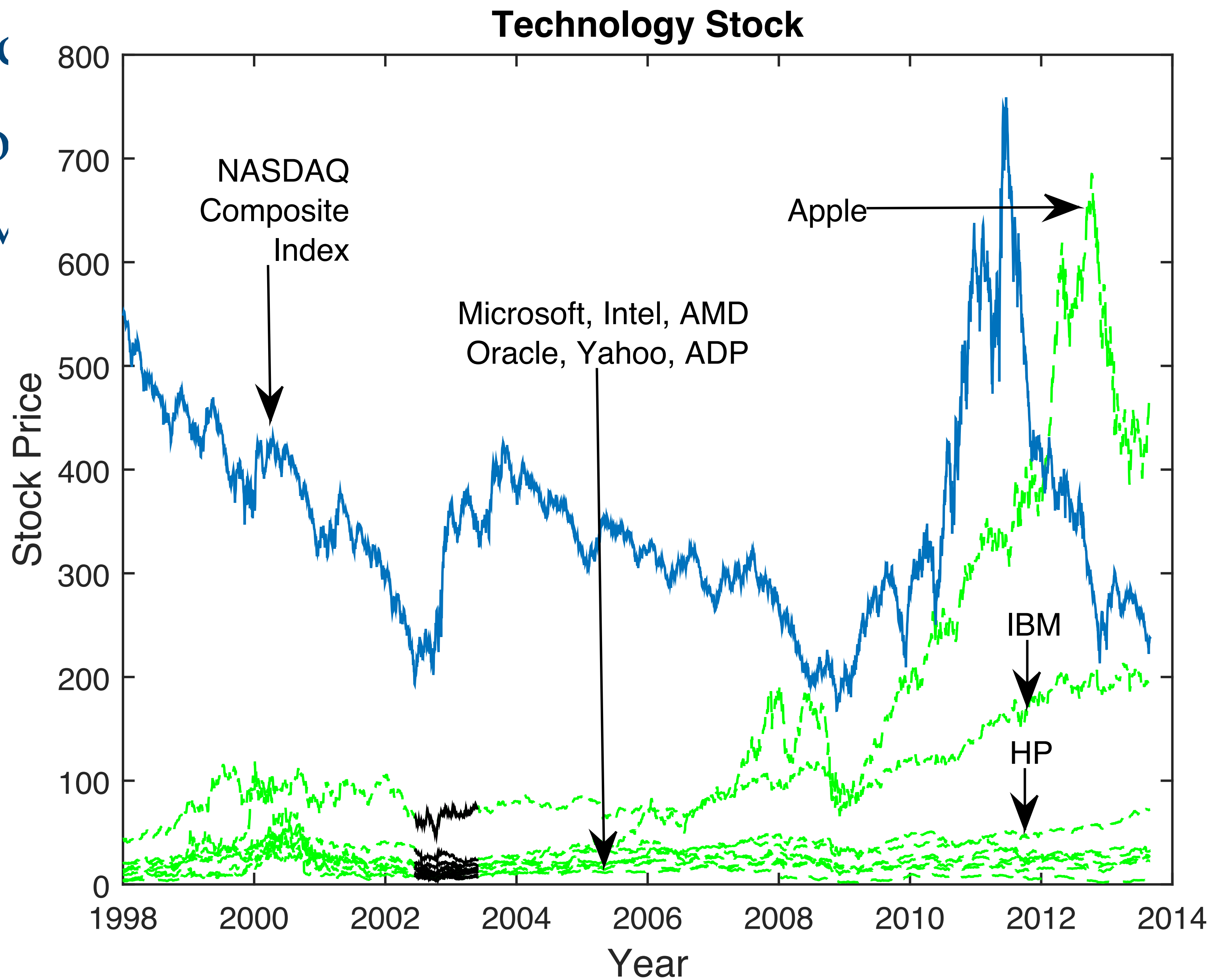

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Example: S & P 500

- Daily trading prices 1998-2013
- 9 tech stocks
  - ADP, AMD, HP, IBM, Intel, Microsoft, Oracle, Yahoo
- Atypical segment in 2003
  - Not clear from stocks themselves
  - Low point of Nasdaq after bubble
    - Perhaps stocks move more in sync?

Conclusion

- We have developed an information theory criterion of atypicality
  - Fundamental
- Works for
  - Discrete valued data
  - Real valued data
- Upper bounded probability of intrinsically atypical data
  - Same for real and discrete case
- Experimental results for stock market data