# Multi-Linear Subspace Estimation and Projection for Efficient RFI Excision in SIMO Systems

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# Agenda



- Motivation
- State-of-the-art
- System model
- **MLSEP**: **M**ulti-**L**inear **S**ubspace **E**stimation and **P**rojection
  - ⇒ Problem setup
  - ⇒ Problem formulation
    - RFI subspace estimation
    - Multi-linear (tensor-based) projection
  - ⇒ MLSEP algorithm
- Simulation results
- Conclusions

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# Motivation



- Radio frequency interference (RFI) is caused by:
  - ⇒ Out-of-band emissions, jamming, spoofing and meaconing
- Such an RFI is prevalent in
  - ⇒ Radio astronomy, microwave radiometry and and global navigation satellite system (GNSS)
- The congestion of licensed spectrum both in satellite and terrestrial communications and the advent of cognitive radios
  - ⇒ Call for **efficient RFI excision (signal processing) algorithms**
- On the other hand, tensor-based parameter estimators based on truncated higher-order SVD (HOSVD) have outperformed their matrix-based counterparts
  - ⇒ Nevertheless, tensors **had never been deployed for RFI excision**

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# State-of-the-art



**RFI detection and excision algorithms**

- **Spectral**
  - ❖Apply DFT and compare with the theoretical threshold
  - ❖Apply FDAF
- **Temporal**
  - ❖Estimation of mean and variance of sample magnitudes
  - ❖RFI detection and excision
- **Spectral-temporal**
  - ❖Time-frequency (TF) repre.
  - ❖RFI estimation and excision
- **Statistical**
  - ❖Apply detection theory
  - ❖Assum. on RFI is needed
- **Spatial filtering-based**
  - ❖RFI subspace estimation
  - ❖orthogonal projection
- **Transformed domain-based**
  - ❖Represent. in other domain
  - ❖RFI detection
  - ❖RFI synthesis and excision

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# System model



- A SIMO system with $N_R$ number of receive antennas suffering from a **stationary severe broadband RFI** (modeled as a zero mean AWGN with variance of $\sigma_f^2$ ) [1]
- The received signal at time n would be

$$\boldsymbol{y}(n) = \sum_{l=0}^{L} \boldsymbol{h}_l s(n-l) + \sum_{l=0}^{L_f} \boldsymbol{g}_l f(n-l) + \boldsymbol{z}(n), \qquad (2)$$

- Assumptions:
  - ⇒ Uncorrelated $s(n)$, $f(n)$ and $\boldsymbol{z}(n)$
  - ⇒ $\boldsymbol{z}(n)$ is a zero mean AWGN vector with a covariance matrix of $\sigma^2 \boldsymbol{I}_{N_R}$
  - ⇒ Perfect estimates of $L$ and $L_f$

[1] M. Wildemeersch and J. Fortuny-Guasch, "Radio frequency interferenceimpact assessment on global navigation satellite systems," EC Joint Research Centre, Security Technology Assessment Unit, Tech. Rep., Jan.2010.

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# MLSEP: Algorithm proposal



- The proposed MLSEP algorithm comprises two phases
  - ⇒ **First phase**: No SOI is transmitted in the first long term interval (LTI)
    - The RFI subspace tensor is estimated using HOSVD
    - From the estimated RFI subspace tensor, the multi-linear projector is derived
  - ⇒ **Second phase**: SOI transmission and RFI excision using the already derived multi-linear projector

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# MLSEP: Problem setup



- Stacking the observation vectors of the $N_R$ receive antennas and $W$ data windows into one highly structured vector of size $N_R.W \times 1$ with respect to the $m$th STI gives

$$\boldsymbol{y}_m = \boldsymbol{H}\boldsymbol{s}_m + \boldsymbol{G}\boldsymbol{f}_m + \boldsymbol{z}_m \in \mathbb{C}^{N_R.W}, \qquad (3)$$

where $\boldsymbol{s}_m = \left[s(mW), \ldots, s(mW - W - L + 1)\right]^T \in \mathbb{C}^{(W+L)}$,

$\boldsymbol{f}_m = \left[f(mW), \ldots, f(mW - W - L_f + 1)\right]^T \in \mathbb{C}^{(W+L_f)}$ and $\boldsymbol{z}_m \in \mathbb{C}^{N_R.W}$ are the SOI, RFI and zero mean AWGN

$\boldsymbol{H} \in \mathbb{C}^{N_R.W \times (W+L)}$ is the SOI filtering matrix as defined in [2]

$\boldsymbol{G} \in \mathbb{C}^{N_R.W \times (W+L_f)}$ is the RFI filtering matrix structured as

$$\boldsymbol{G} = \left[\boldsymbol{G}_1^T, \boldsymbol{G}_2^T, \ldots, \boldsymbol{G}_{N_R}^T\right]^T, \qquad (4)$$

$$\boldsymbol{G}_j = \begin{bmatrix} g_j^0 & \ldots & g_j^{L_f} & 0 & \ldots & \ldots & 0 \\ 0 & g_j^0 & \ldots & g_j^{L_f} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & g_j^0 & \ldots & g_j^{L_f} \end{bmatrix}. \qquad (5)$$

# MLSEP: Problem setup (cont.)



- The horizontal concatenation of $N$ $\boldsymbol{y}_m$s in (3) renders

$$\boldsymbol{Y} = \boldsymbol{HS} + \boldsymbol{GF} + \boldsymbol{Z} \in \mathbb{C}^{N_R.W \times N}. \qquad (6)$$

- In the first LTI, no SOI transmission occurs and the received signal would then be

$$\boldsymbol{Y}_I = \boldsymbol{GF} + \boldsymbol{Z} \in \mathbb{C}^{N_R.W \times N}. \qquad (7)$$

- From (7), the RFI subspace $\hat{U}_I \in \mathbb{C}^{N_R.W \times (W+L_f)}$ estimated using SVD as

$$\boldsymbol{Y}_I = [\hat{U}_I \, \hat{U}_n] \begin{bmatrix} \hat{\boldsymbol{\Sigma}}_I & \boldsymbol{0}_{r \times (N-r)} \\ \boldsymbol{0}_{(N_R.W-r) \times r} & \hat{\boldsymbol{\Sigma}}_n \end{bmatrix} [\hat{\boldsymbol{V}}_I \, \hat{\boldsymbol{V}}_n]^H, \qquad (8)$$

where $\hat{\Sigma}_I = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ and $r = W + L_f$.

[2] B. Song, F. Roemer, and M. Haardt, "Blind estimation of simo channels using a tensor-based subspace method," in Signals, Systems and Computers (ASILOMAR), 2010 Conf. Record of the 44th Asilomar Conf. on, Nov 2010, pp. 8–12.

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# MLSEP: Formulation



- To deploy the **inherent structure of the measurement data**, we model the received signal by a 3-way tensor $\mathcal{Y} \in \mathbb{C}^{N_R \times W \times N}$, where $N_R$, $W$ & $N$ are the number of antennas, samples per a STI and non-overlapping STIs per LTI, respectively

- If $[\mathcal{Y}]_{(3)}^{T}$ should be equal to $\boldsymbol{Y}$ in (6), the multi-linear equivalent of (6) would be

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}} \times_3 \boldsymbol{S}^T + \boldsymbol{\mathcal{G}} \times_3 \boldsymbol{F}^T + \boldsymbol{\mathcal{Z}}, \qquad (9)$$

where $\boldsymbol{\mathcal{H}} \in \mathbb{C}^{N_R \times W \times (W+L)}$ and $\boldsymbol{\mathcal{G}} \in \mathbb{C}^{N_R \times W \times (W+L_f)}$ are constructed as in Fig. 1 and $\boldsymbol{\mathcal{Z}}$ is the noise tensor

- Besides, $[\boldsymbol{\mathcal{H}}]_{(3)}^{\mathrm{T}} = \boldsymbol{H}$ and $[\boldsymbol{\mathcal{G}}]_{(3)}^{\mathrm{T}} = \boldsymbol{G}$

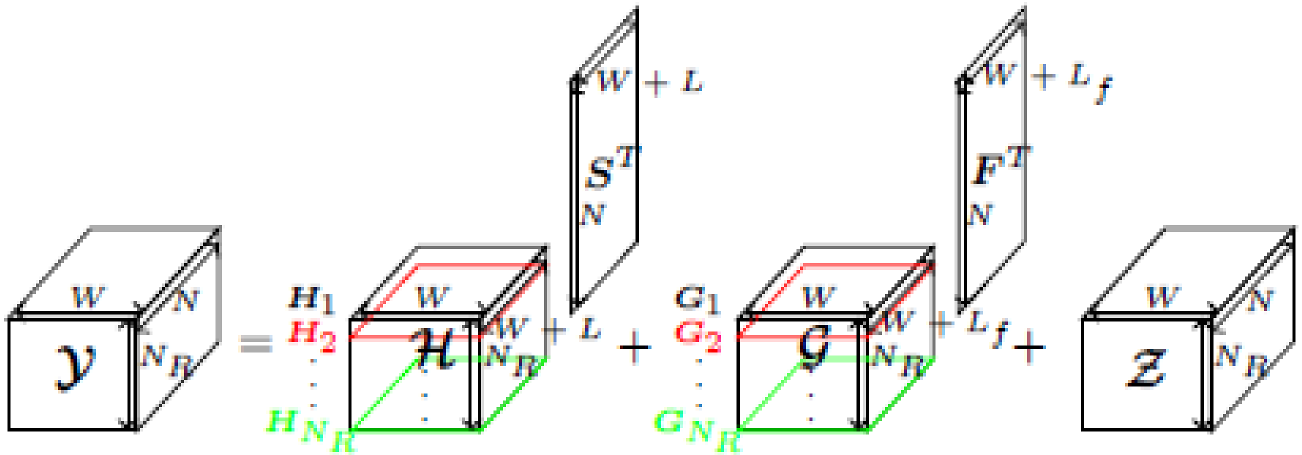


Fig. 1. Multi-linear formulation from the received signal per LTI $\boldsymbol{Y}$ in (6).

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- In the first LTI, no SOI is transmitted and hence the truncated HOSVD of the received signal $\mathcal{Y}_I = \mathcal{G} \times_3 \boldsymbol{F}^T + \mathcal{Z}$ would be

$$\mathcal{Y}_I \approx \hat{\mathcal{S}}^{[I]} \times_1 \hat{\boldsymbol{U}}_1^{[I]} \times_2 \hat{\boldsymbol{U}}_2^{[I]} \times_3 \hat{\boldsymbol{U}}_3^{[I]}, \qquad (10)$$

where $\hat{\mathcal{S}}^{[I]} \in \mathbb{C}^{r_1 \times r_2 \times r_3}$ is a core-tensor which satisfies the all-orthogonality conditions, $r_n$ is the $n$-rank of the noiseless tensor $\tilde{\mathcal{Y}}_I = \mathcal{G} \times_3 \boldsymbol{F}^T$

$$r_1 = \min(N_R,\ L_f + 1),\ r_2 = \min(W,\ N.N_R)\ \&\ r_3 = \min(N,\ W + L_f).$$

# RFI signal subspace estimation (cont.)

- From (10), the estimated RFI signal subspace tensor is defined as [3]

$$\hat{\mathcal{U}}^{[I]} = \hat{\mathcal{S}}^{[I]} \times_1 \hat{U}_1^{[I]} \times_2 \hat{U}_2^{[I]} \times_3 \hat{\Sigma}_I^{-1}. \qquad (11)$$

- Note that $\left[\hat{\mathcal{U}}^{[I]}\right]_{(3)}^T \in \mathbb{C}^{N_R.W \times r_3}$ span the estimated RFI signal subspace and inspires the underneath theorem

*Theorem 1:* The tensor-based RFI subspace estimator $\left[\hat{\mathcal{U}}^{[I]}\right]_{(3)}^T$ and the matrix-based RFI subspace estimator $\hat{U}_I$ are related by

$$\left[\hat{\mathcal{U}}^{[I]}\right]_{(3)}^T = (\hat{T}_1 \otimes \hat{T}_2) \cdot \hat{U}_I, \qquad (12)$$

where $\hat{T}_r = \hat{U}_r^{[I]} \cdot \hat{U}_r^{[I]^H}$, $r = 1, 2$.

[3] F. Roemer, M. Haardt, and G. Del Galdo, “Analytical performance assessment of multi-dimensional matrix- and tensor-based esprit-type algorithms,” IEEE Trans. Signal Process., vol. 62, no. 10, pp. 2611–2625, May 2014.

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# Multi-linear projection



- The multi-linear projector for **perfect excision** is stated in the underneath theorem

*Theorem 2:* For a perfect $\mathcal{U}^{[I]}$, the multi-linear projector $\mathcal{P} \in \mathbb{C}^{N_R \times W \times N_R.W}$ which evokes perfect RFI excision is given by

$$\mathcal{P} = \mathcal{I}_3 - \hat{\mathcal{U}}^{[I]} \times_3 \left(\hat{\mathcal{U}}^{[I]}\right)^{+3}, \quad (13)$$

where $\mathcal{I}_3 \in \mathbb{C}^{N_R \times W \times N_R.W}$ is the 3-mode identity tensor, $\left(\hat{\mathcal{U}}^{[I]}\right)^{+3}$ is the 3-mode pseudo-inverse tensor, $[\mathcal{I}_3]_{(3)} = \boldsymbol{I}_{N_R.W}$ and $\left[\left(\hat{\mathcal{U}}^{[I]}\right)^{+3}\right]_{(3)} = \left[\hat{\mathcal{U}}^{[I]}\right]^{+}_{(3)}$.

- However, perfect excision is impossible and we define the root mean square excision error (RMSEE) as

$$\text{RMSEE} = \sqrt{\mathbb{E}\left\{\|\boldsymbol{P}\boldsymbol{G}\|_F^2\right\}} \quad (14)$$

$$\text{RMSEE} = \sqrt{\mathbb{E}\left\{\|[\mathcal{P} \times_3 \mathcal{G}]_{(3)}^T\|_F^2\right\}}. \quad (15)$$

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# The MLSEP algorithm



**Algorithm I**: MLSEP for efficient RFI excision in SIMO systems

**Input**: $Y_I$, $Y$, $N_R$, $W$, $L$, $L_f$, $N$

**Assumptions**: $N \succeq \{W+L,\, W+L_f\}$, $W \succ \{L,\, L_f\}$.

**Initialization**: $r_1 = \min(N_R,\, L_f+1)$, $r_2 = \min(W,\, N.N_R)$

$\mathcal{Y}_I$ =the tensorization of $\left[\mathcal{Y}_I\right]_{(3)} = Y_I^T$

$[\mathcal{Y}_I]_{(1)} = U_1 \Sigma_1 V_1^H$; $\hat{U}_1^{[I]} = U_1(:, 1:r_1)$

$[\mathcal{Y}_I]_{(2)} = U_2 \Sigma_2 V_2^H$; $\hat{U}_2^{[I]} = U_2(:, 1:r_2)$

$Y_I = U \Sigma V^H$; $\hat{U}_I = U(:, 1: W+L_f)$

$\left[\hat{\mathcal{U}}^{[I]}\right]_{(3)}^T = \left(\hat{T}_1 \otimes \hat{T}_2\right).\hat{U}_I$; $\hat{T}_r = \hat{U}_r^{[I]}.\hat{U}_r^{[I]^H}$, $r=1,2$

$\hat{\mathcal{U}}^{[I]}$ = the tensorization of $\left[\hat{\mathcal{U}}^{[I]}\right]_{(3)}$

$\mathcal{P} = \mathcal{I}_3 - \hat{\mathcal{U}}^{[I]} \times_3 \left(\hat{\mathcal{U}}^{[I]}\right)^{+3} \in \mathbb{C}^{N_R \times W \times N_R.W}$

**Repeat**

$\mathcal{Y}$ =the tensorization of $\left[\mathcal{Y}\right]_{(3)} = Y^T$

**return** $\left[\mathcal{P} \times_3 \mathcal{Y}\right]_{(3)}^T$

**Until** no SOI transmission

**Multi-linear RFI subspace estimation**

**Multi-linear projection**

**Multi-linear RFI excision**

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# Simulation results



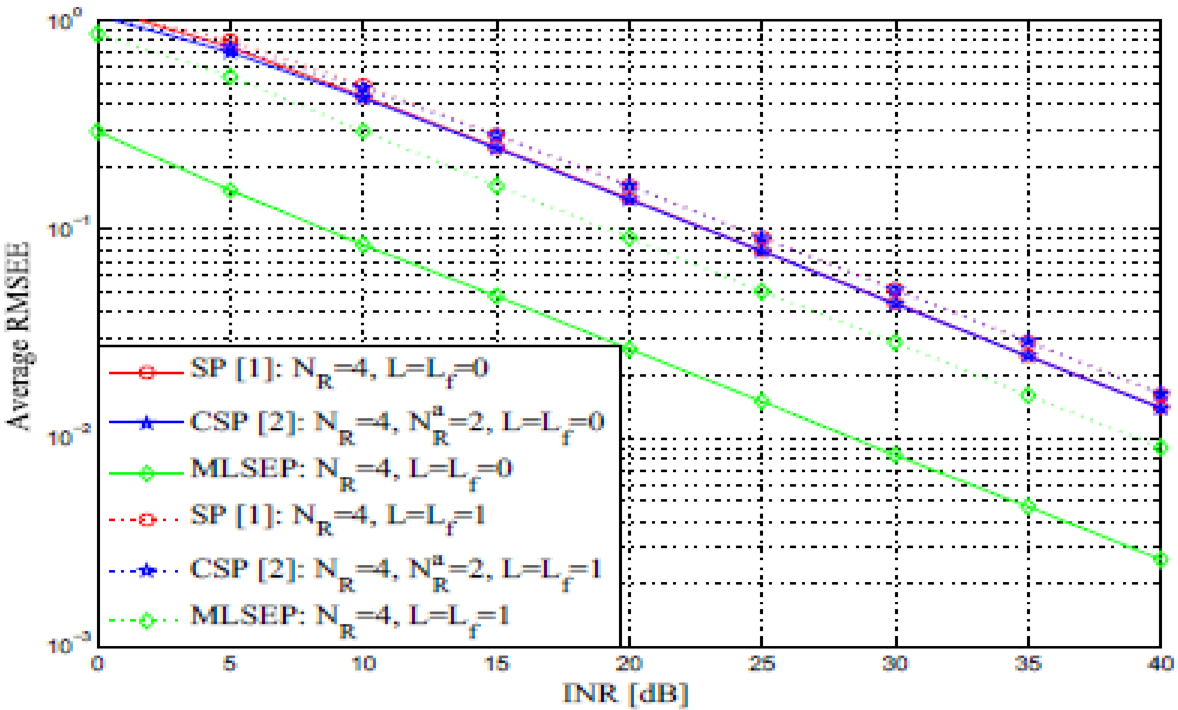


Fig. 2. Average RMSEE for an RFI excision using SP, CSP and MLSEP during $N_{SOI} = 200$ LTIs and 40 observed symbols per LTI at $W = 4$, $N = 10$, $\alpha = 100$ and a pre-excision SINR of 0 dB.

| Algorithms | Single-path ($L+1=L_f+1=1$) scenario | | | | | Multi-path ($L+1=L_f+1=2$) scenario | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | INR [dB] | | | | | INR [dB] | | | | |
| | 0 | 10 | 20 | 30 | 40 | 0 | 10 | 20 | 30 | 40 |
| Perfect excision | 2.80 dB | 10.21 dB | 19.84 dB | 29.80 dB | 39.80 dB | 2.92 dB | 10.35 dB | 19.95 dB | 29.91 dB | 39.92 dB |
| SP [1] | 1.35 dB | 8.11 dB | 17.67 dB | 27.62 dB | 37.61 dB | 1.31 dB | 7.62 dB | 17.02 dB | 26.96 dB | 36.96 dB |
| CSP [2] | 1.55 dB | 8.18 dB | 17.66 dB | 27.63 dB | 37. 65 dB | 1.48 dB | 7.76 dB | 17.09 dB | 27.01 dB | 36.99 dB |
| MLSEP | 2.67 dB | 10.11 dB | 19.74 dB | 29.69 dB | 39.69 dB | 1.87 dB | 9.12 dB | 18.80 dB | 28.76 dB | 38.76 |

TABLE I

AVERAGE SINR GAIN [DB] EVOKED BY PERFECT EXCISION, SP [1], CSP [2] AND MLSEP FOR BOTH SINGLE-PATH AND MULTI-PATH SCENARIOS DURING $N_{SOI}=200$ LTIS AND 40 OBSERVED SYMBOLS PER LTI AT $W=4$, $N=10$, $\alpha=100$ AND A PRE-EXCISION SINR OF 0 DB.

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# Conclusions and outlooks



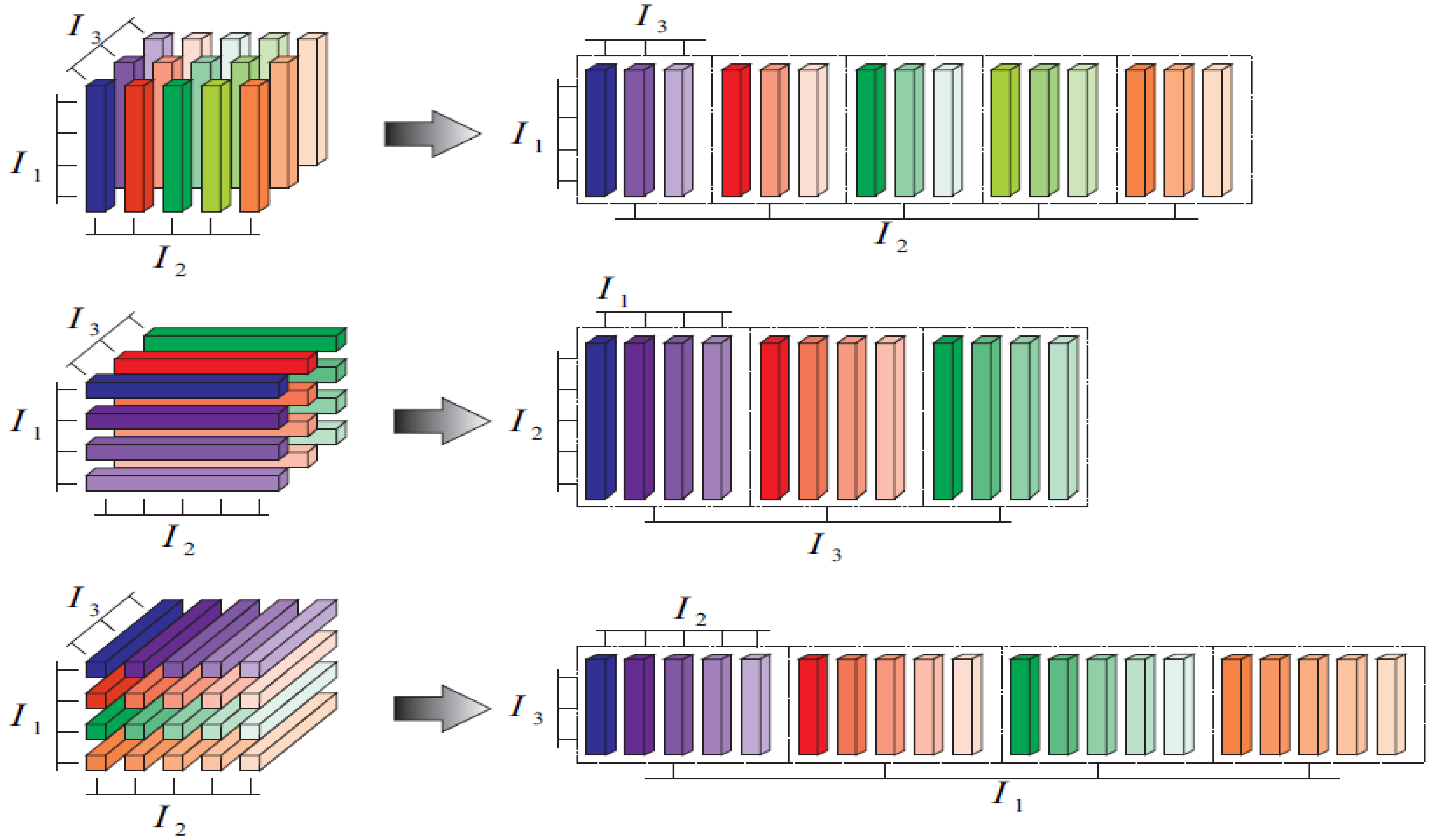
- The paper introduces the multi-linear algebra framework to the RFI excision research

- The MLSEP algorithm outperforms the state-of-the-art projection-based RFI excision algorithms algorithms

# Backups

# Unfoldings of a 4x5x3 tensor in reverse cyclical column ordering [6]



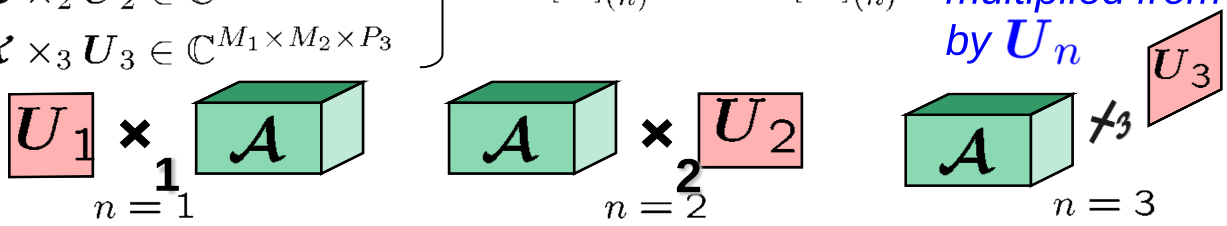


[3] F. Roemer, "Advanced algebraic concepts for efficient multi-channel signal processing," Ph.D. dissertation, Ilmenau Univ. of Tech., 2012.

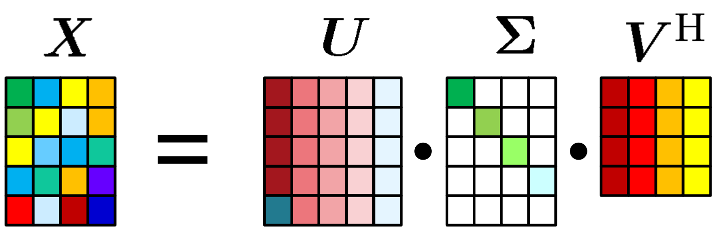

- *n*-mode products between $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ and $\boldsymbol{U}_n \in \mathbb{C}^{P_n \times M_n}$

$$\left.\begin{aligned} \boldsymbol{\mathcal{Y}} &= \boldsymbol{\mathcal{X}} \times_1 \boldsymbol{U}_1 \in \mathbb{C}^{P_1 \times M_2 \times M_3} \\ \boldsymbol{\mathcal{Y}} &= \boldsymbol{\mathcal{X}} \times_2 \boldsymbol{U}_2 \in \mathbb{C}^{M_1 \times P_2 \times M_3} \\ \boldsymbol{\mathcal{Y}} &= \boldsymbol{\mathcal{X}} \times_3 \boldsymbol{U}_3 \in \mathbb{C}^{M_1 \times M_2 \times P_3} \end{aligned}\right\} \Leftrightarrow [\boldsymbol{\mathcal{Y}}]_{(n)} = \boldsymbol{U}_n \cdot [\boldsymbol{\mathcal{X}}]_{(n)}$$

*i.e., all the n-mode vectors multiplied from the left-hand-side by* $\boldsymbol{U}_n$



## Matrices



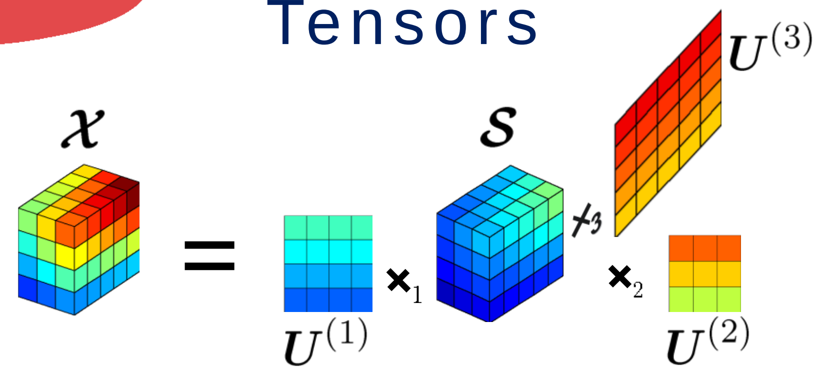

$\boldsymbol{U}, \boldsymbol{V}$ – unitary

$\boldsymbol{\Sigma}$ – diagonal

$\boldsymbol{\Sigma}$ – "all orthogonal"

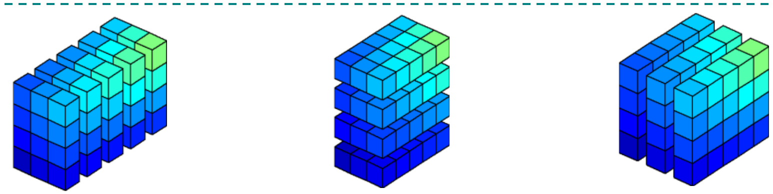

[4] F. Roemer, "Advanced algebraic concepts for efficient multi-channel signal processing," Ph.D. dissertation, Ilmenau Univ. of Tech., 2012.


## Tensors




$\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \boldsymbol{U}^{(3)}$ – unitary

$\boldsymbol{\mathcal{S}}$ – full tensor




"all orthogonality" of $\boldsymbol{\mathcal{S}}$

$[\boldsymbol{\mathcal{S}}]_{(n)}$ – have orthogonal rows

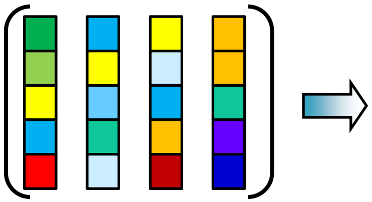
$$[\boldsymbol{\mathcal{S}}]_{(n)} \cdot [\boldsymbol{\mathcal{S}}]_{(n)}^{\mathrm{H}} = \boldsymbol{\Sigma}^{(n)^2}$$

# The "$n$-rank" of a tensor [4]



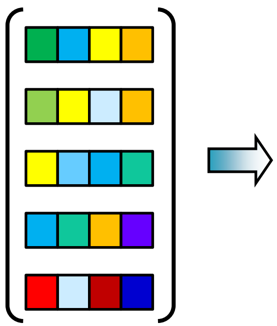
## Matrices

$\text{rank}\{\boldsymbol{X}\}$ - column rank

$\boldsymbol{X} =$ 

# linearly independent column vectors

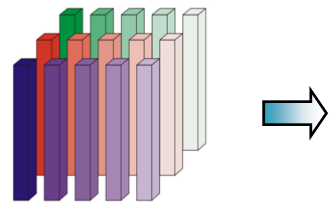
$\text{rank}\{\boldsymbol{X}^{\text{T}}\}$ - row rank

$\boldsymbol{X} =$ 

# linearly independent row vectors

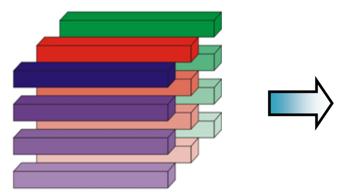
## Tensors

$1\text{-rank}\{\boldsymbol{\mathcal{X}}\} = \text{rank}\{[\boldsymbol{\mathcal{X}}]_{(1)}\}$

$\boldsymbol{\mathcal{X}} =$ 

# linearly independent 1-mode vectors

$2\text{-rank}\{\boldsymbol{\mathcal{X}}\} = \text{rank}\{[\boldsymbol{\mathcal{X}}]_{(2)}\}$

$\boldsymbol{\mathcal{X}} =$ 

# linearly independent 2-mode vectors

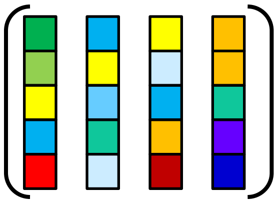
[4] F. Roemer, "Advanced algebraic concepts for efficient multi-channel signal processing," Ph.D. dissertation, Ilmenau Univ. of Tech., 2012.

# The “$n$-rank” of a tensor [4]



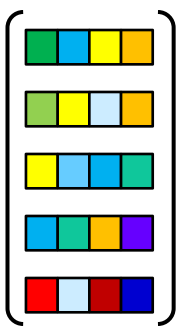
## Matrices

rank$\{\boldsymbol{X}\}$ - column rank

$\boldsymbol{X} =$ 

# linearly independent column vectors

rank$\{\boldsymbol{X}^{\mathrm{T}}\}$ - row rank

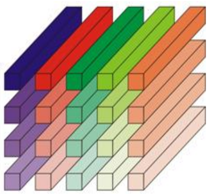
$\boldsymbol{X} =$ 

# linearly independent row vectors

$$\text{rank}\{\boldsymbol{X}\} = \text{rank}\{\boldsymbol{X}^{\mathrm{T}}\}$$
$$\text{rank}\{\boldsymbol{X}\} \leq \min\{M_1, M_2\}$$

## Tensors

$$3\text{-rank}\{\boldsymbol{\mathcal{X}}\} = \text{rank}\{[\boldsymbol{\mathcal{X}}]_{(3)}\}$$

$\boldsymbol{\mathcal{X}} =$ 

# linearly independent 3-mode vectors

$$1\text{-rank}\{\boldsymbol{\mathcal{X}}\} \neq 2\text{-rank}\{\boldsymbol{\mathcal{X}}\} \neq 3\text{-rank}\{\boldsymbol{\mathcal{X}}\}$$
$$n\text{-rank}\{\boldsymbol{\mathcal{X}}\} \leq M_n$$