Fast Computation of Generalized Waterfilling Problems

Presented by

Kalpana

co-author : Mohammed Zafar Ali Khan

Department of Electrical Engineering
Indian Institute of Technology Hyderabad

Dec 14, 2015

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Outline

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Introduction

Introduction

- What is waterfilling?
- Waterfilling problem (WFP) allocates powers to the resources of the transmitting user. These allocated powers maximize the transmitting user's capacity while following the total power budget.
- The resource's allocated power is inversely proportional to the noise level of the resource in WFP.
- Resources are the sub-carriers in Orthogonal Frequency Division Multiplexing (OFDM) or the normal frequency bands or the usage of the same sub-carriers in different time slots.
- WFP finds applications in various fields of communication systems.
- What is the challenge?
- It is a nonlinear problem and fast computation will make implementation easy.

Introduction...

- Traditionally, the problem is solved iteratively [1].
- However Number of Iterations is not known apriori
- Computationally complex and results in loss of capacity.
- Of late 'exact'(fixed computational complexity) algorithms have been developed to solve this problem [2], [3]
- These work only with the Traditional Water Filling (TWF) and it's minor variants.

---

[1] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines," IEEE J. Sel. Areas Commun., vol. 20, no. 5, Jun. 2002

[2] D. P. Palomar and J. R. Fonollosa, "Practical algorithms for a family of waterfilling solutions," IEEE Trans. Signal Process., vol. 53, no. 2, pp. 686695, Feb. 2005 ;

[3] P. He, L. Zhao, S. Zhou, and Z. Niu, "Water-filling: A geometric approach and its application to solve generalized radio resource allocation problems," IEEE Trans. Wireless Commun., vol. 12, no. 7, Jul. 2013.

Introduction...

Traditional waterfilling problem (TWF) is described as

$$\max_{P_i} \; C = \sum_{i=1}^{M} \log_2 \left(1 + \frac{P_i}{N_i}\right)$$

$$\text{with constraints} : \sum_{i=1}^{M} P_i \leq P_t,$$

$$P_i \geq 0, 1 \leq i \leq M; \tag{1}$$

- where power budget is P_t
- sequence $\{N_i\}_{i=1}^{M}$, corresponding to M user resources/subchannels
- TWF does not have weights.

Introduction...

- The solution to (1) is obtained by using KKT conditions as

$$P_i = \left(\frac{1}{\lambda} - N_i\right)^+ ; i = 1, \cdots, M;$$
$$\sum_{i=1}^{M} P_i \leq P_t \tag{2}$$

where

- $A^+ \triangleq \max(A, 0)$,
- λ is the Lagrangian and $\frac{1}{\lambda}$ indicates the 'water level'.
- Substituting P_i in the sum power constraint, we can solve for λ; provided we know the i's for which P_i is positive.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

The Generalized Waterfilling Problem (GWFP)

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

The Generalized Waterfilling Problem (GWFP)

- The GWFP is described by

$$\max_{P_i} \; C = \sum_{i=1}^{M} w_i \log_2\left(1 + \frac{P_i}{N_i}\right) \tag{3}$$

with constraints:

$$\sum_{i=1}^{M} x_i P_i \leq P_t \tag{4}$$

$$\& P_i \geq 0, i \leq M. \tag{5}$$

where x_i, w_i are corresponding weights.

GWFP ...

GWFP is used to solve a wide family of WaterFilling Problems :

- $w_i = x_i = 1$ converts GWFP to TWF
- $x_i = 1 \Rightarrow$ MAC scheduling with weights w_i representing the priorities of the users or the length of the queues of the users
- $w_i = 1 \Rightarrow$ In Cognitive Radio, x_i is the i^{th} subcarrier gain from the secondary user to the primary.
- $w_i \neq 1$ & $x_i \neq 1 \Rightarrow$ In downlink, w_i's are the priorities of the users and x_i's are the gains from Base station to the user.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Equivalence based Waterfilling (EBWF)

EBWF...

- For convenience of presentation, the case where $x_i = 1, \ \forall i$ is called the **Weighted Waterfilling Problem (WWFP)**.
- Solution of (3)-(5) (or GWFP) is given by

$$P_i = \frac{w_i}{x_i}\left(\frac{1}{\lambda} - \bar{N}_i\right)^+ ; i = 1, \cdots, M;$$
$$\sum_{i=1}^{M} x_i P_i \leq P_t \tag{6}$$

where

- $A^+ \triangleq \max(A, 0)$, λ is the Lagrangian and
- $\frac{1}{\lambda}$ indicates the 'water level' of the GWFP.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

EBWF...

Lemma

For every GWFP, defined in (3)-(5), there exists a WWFP of the form

$$\max_{P_i} \; C = \sum_{i=1}^{M} \tilde{w}_i \log_2\left(1 + \frac{P_i}{N_i}\right)$$
$$\text{with constraints :}$$
$$\sum_{i=1}^{M} P_i \leq P_t, \;\; P_i \geq 0, 1 \leq i \leq M; \tag{7}$$

where $\tilde{w}_i = w_i / x_i \; \forall i$. Moreover, the WWFP has the same solution as the GWFP.

- In what follows we assume a WWFP only.

Preliminaries

Definition (The Number of Positive Powers, K)

Let $\mathcal{I} = \{i; \ni P_i > 0\}$ be the set of indices where P_i is positive. Then the number of positive powers, $K = |\mathcal{I}|$, is the cardinality of the set, $\mathcal{I}$.

- Without loss of generality, **we assume that $\bar{N}_i$'s are sorted in ascending order**.
- It follows that, P_i; $i = 1, \cdots, M$ are in descending order and the first K powers are positive. Accordingly,

$$P_i = \begin{cases} \tilde{w}_i \left[\frac{1}{\lambda} - \bar{N}_i\right], & 1 \le i \le K; \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

- From the weighted sum power constraint (6) , we have

$$\frac{K}{\lambda} = P_t + \sum_{i=1}^{K} \bar{N}_i. \quad (9)$$

Concept of Equivalence

- Note that in (9) the unknowns are on the LHS and depend on the power budget, P_t and weighted sum $\sum_{i=1}^{K} \bar{N}_i$.
- Note that the nonlinearity is due to $\sum_{i=1}^{K} \bar{N}_i$.
- Observe that two different GWFP's with different $\bar{N}_i$'s but have the same weighted sum $\sum_{i=1}^{K} \bar{N}_i$ will have the same K.

Definition (l-Equivalent Waterfilling Problems)

*For a given l, two WFPs are said to be l-***Equivalent Waterfilling Problems** *if*

1. *They have the same weighted sum,* $\Psi(l) = \sum_{i=1}^{l} \bar{N}_i = \sum_{i=1}^{l} \frac{N_i}{w_i}$,
2. *The l-th step $\bar{N}_l$ is same.*

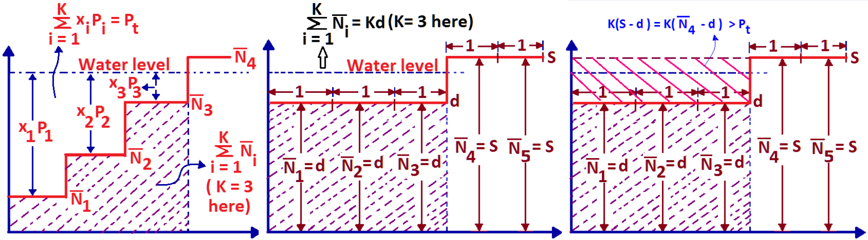
i.e. the total 'water' contained in l-equivalent waterfilling problems is same in the l-th step.

Examples of I-Equivalent WFPs

Proposition

For every WWFP constructed, the following inequalities hold:

$$i)\sum_{m=1}^{K-1}(\bar{N}_K-\bar{N}_m)<P_t\ \&\ ii)\sum_{m=1}^{K}(\bar{N}_{K+1}-\bar{N}_m)\geq P_t \qquad (10)$$



Given WWFP (b) 1-step WFP (c) 1-step WFP area

Equal 'sum of noise levels' (slanted dashes) for both Fig. (a) & Fig. (b) (K=3 in figure).

1-Step WFP

1-Step WFP :

- $w_i = x_i = 1$ and $\bar{N}_i = d, i \leq m$ and $\bar{N}_i = s, i \geq m$ with $s > d$.
- $$\Psi_S(l) = \sum_{i=1}^{l} \bar{N}_i = \begin{cases} ld, & l \leq m, \\ ld + s(l-m), & otherwise. \end{cases} \quad (11)$$

In 1-Step WFP, K takes one of the two values :

$$K = \begin{cases} m, & (s-d)m \geq P_t; \rightarrow ms \geq P_t + \sum_{i=1}^{m} \bar{N}_i \rightarrow ms \geq d_m \\ M, & otherwise. \end{cases} \quad (12)$$

- This WFP does not occur in practical scenarios, but we will use it to obtain a solution for practical WWFP's.
- 1-step is a WFP with a closed form solution for K. We can define more examples by assuming a structure on the $\bar{N}_i$'s.

General l-Equivalence based Waterfilling for obtaining K

Require: Inputs required are M, P_t, $\bar{N}_i$ (in ascending order).
Ensure: Output is K.

1: Let $m = 1$. Construct an equivalent WFP (Arithemtic or Geometric or 1–step) for m.
2: Find the K for the equivalent WFP denoted as K_{eq} .
3: **if** $K_{eq} == m$ **then**
4: $K = K_{eq}$; Exit the algorithm.
5: **else**
6: increment m and go to 2.
7: **end if**

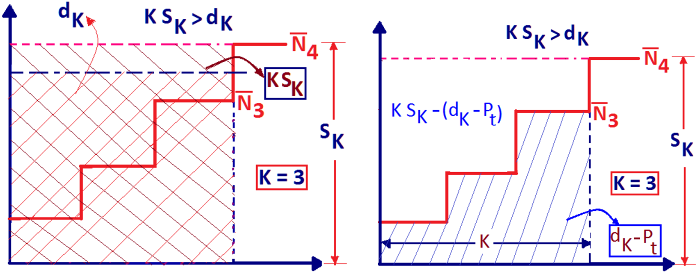
- This algorithm gives the general setup for obtaining algorithms for solving GWFP based on a WFP with known K
- We now give the detailed implementation of EBWF using 1-step WFP

EBWF using 1-step WFP

Require: Inputs required are M, P_t, $\bar{N}_i$ (in ascending order).
Ensure: Output is K.

1: $i = 1$. Denote $d_0 = P_t$.
2: Calculate $d_i = d_{i-1} + \bar{N}_i$.
3: Calculate s, the second step, of the 1–step l-equivalent WFP for each i (denoted by s_i) as $s_i = N_{i+1}$
4: **if** $is_i > d_i$ **then**
5: $K \leftarrow i$. Exit the algorithm.
6: **else**
7: $i \leftarrow$ i+1, Go to 2
8: **end if**

EBWF Geometric Interpretation



This algorithm lends itself to a nice geometric interpretation:

- the term is_i gives the total area of i steps;
- d_i gives the area of the i 'Noise' steps along with P_t
- the difference, $is_i - (d_i - P_t)$, is the area where the 'water can be poured'
- If $is_i > d_i$, then the number of steps is 'enough' to store all the water.

Calculating the Powers and Computational Complexity

- Calculate the water level as $\frac{1}{\lambda} = \frac{d_K}{K}$.
- Calculate K powers using the water level & (6).

Table: Computational Complexity of various known solutions to WWFP

No. of flops in PWFA	**No. of flops in IWFA**	**No. of flops in algo. of [6]**	**No. of flops in algo. of [7]**	**No. of flops in GWF of [5]**	**No. of flops in proposed solution**
(no. of iterations)× O(M^2) [3]	(no. of iterations) × (5M + 1) + 2M	$\frac{(M-K+7)(M+K)}{2}$	$\frac{M(M+3)}{2}$ + 2K(2K+1)	8M + 3[5]	4(M+1)

- Observe the reduction by a factor of 2

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Simulation Results

Table: Computational complexities of existing and the proposed solution for WWFP, $P_t = 1, \sigma^2 = 10^{-1}$. h_i, w_i & x_i are exponentially distributed with variance 1

M → K	**No. of flops in PWFA**	**No. of flops in IWFA**	**No. of flops in [6]**	**No. of flops in [7]**	**No. of flops in GWF**	**No. of flops in proposed solution**
128 → 27	40599552 (2478)	28460 (44)	8370	11354	1027	161
192 → 32	136691712 (3708)	44590 (46)	18704	22880	1539	191
256 → 42	236781568 (3613)	58157 (45)	32929	40292	2051	251
512 → 53	1.2336e+009 (4706)	121391 (47)	131645	143182	4099	317
1024 → 73	7.3725e+009 (7031)	268340 (52)	525463	547286	8195	437

- Number of flops in IWFA, algorithms of [6] & [7] $> O(10^2) \times$ number of flops in PS. (PS → proposed solution)
- Number of flops in GWF of [5] $> O(10) \times$ number of flops in PS.

Conclusion

Conclusion

1. We have proposed solutions for doing power allocation to Generalized Water filling problem namely GWFP.
2. The proposed solutions produce optimal powers.
3. Also, the number of flops for GWFP are of O(M) and are far less than the computation complexities of existing algorithms.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

References

References I

[1] M. Kobayashi and G. Caire, "Iterative waterfilling for weighted rate sum maximization in mimo-ofdm broadcast channels," in *Proc. of ICASSP 2007*, Apr. 2007, pp. III–5 – III–8.

[2] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE J. Sel. Areas Commun.*, vol. 20, no. 5, Jun. 2002.

[3] A. Liu, Y. Liu, V. K. N. Lau, H. Xiang, and W. Luo. (2011) Polite water-filling for Weighted sum-rate maximization in MIMO B-MAC networks under Multiple Linear Constraints. [Online]. Available: http://ecee.colorado.edu/~liue/publications/index.html

[4] N.Kalpana, M. Z. A. Khan, and U.B.Desai, "Optimal Power allocation for Secondary users in CR networks," in *Proc. of 2011 IEEE Advanced Networking and Telecommunication Systems (ANTS) Conference (IEEE ANTS 2011),Bangalore, India*, Dec. 2011.

[5] P. He, L. Zhao, S. Zhou, and Z. Niu, "Water-filling: A geometric approach and its application to solve generalized radio resource allocation problems," *IEEE Trans. Wireless Commun.*, vol. 12, no. 7, Jul. 2013.

[6] D. P. Palomar and J. R. Fonollosa, "Practical algorithms for a family of waterfilling solutions," *IEEE Trans. Signal Process.*, vol. 53, no. 2, pp. 686–695, Feb. 2005.

[7] E. Altman, K. Avrachenkov, and A. Garnaev, "Closed form solutions for water-filling problems in optimization and game frameworks," *Telecommunication Systems*, vol. 47, no. 1-2, pp. 153–164, 2011.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Thank you....

????