# Efficient Single/Multiple Unimodular Waveform Design With Low Weighted Correlations


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## Motivations

- Waveform design approaches *manipulate with correlation properties*. Focuses: 1) waveform quality itself; 2) mismatched filter design.
- Corresponding problems are *non-convex* and can grow to *large scale* as the code length and number of waveforms increase.
- The difficulty is also that the *required accuracy is high*.

## Contributions

- Fast and efficient algorithm utilizing inherent algebraic structures in weighted integrated sidelobe level (WISL) expressions.
- Deriving the objective into an alternative quartic form which allows to apply the quartic-quadratic transformation.
- The non-convex quartic optimization into quadratic form by means of majorization-minimization (MaMi).
- Proposed algorithm shows faster convergence and better correlation properties compared to its conterparts.

## WISL and Problem Formulation

- $\mathbf{Y}$: $P \times M$ waveform matrix ($M$: waveform number; $P$: code length).
- WISL expression:

$$\zeta = \sum_{m=1}^{M} \sum_{\substack{p=-P+1 \\ p\neq 0}}^{P-1} \gamma_p^2 |r_{mm}(p)|^2 + \sum_{m=1}^{M} \sum_{\substack{m'=1 \\ m'\neq m}}^{M} \sum_{p=-P+1}^{P-1} \gamma_p^2 |r_{mm'}(p)|^2 .$$

- Unimodular waveform design problem:

$$\min_{\mathbf{y}} \zeta \quad \text{s.t.} \ |y_m(p)| = 1, \ m = 1, \ldots, M; \ p = 1, \ldots, P.$$

## Main Results

- Transforming into frequency domain and performing derivations:

$$\zeta = \frac{1}{2P} \sum_{p=1}^{2P} \left\| \mathbf{Y}^{\mathrm{H}} \left( (\mathbf{a}_p \mathbf{a}_p^{\mathrm{H}}) \odot \mathbf{\Gamma} \right) \mathbf{Y} - \gamma_0 P \mathbf{I}_M \right\|^2 .$$

- Alternative objective function in quartic form:

$$\zeta = \sum_{p=1}^{2P} \sum_{k=1}^{K} \sum_{k'=1}^{K} \left( \mathbf{y}^{\mathrm{H}} \Big( (\mathbf{A}_p \mathbf{A}_p^{\mathrm{H}}) \odot \mathbf{\Gamma}_{kk'}^{\mathrm{real}} \Big) \mathbf{y} \right)^2 + \left( \mathbf{y}^{\mathrm{H}} \Big( (\mathbf{A}_p \mathbf{A}_p^{\mathrm{H}}) \odot \mathbf{\Gamma}_{kk'}^{\mathrm{img}} \Big) \mathbf{y} \right)^2 .$$

- Generalized majorization function for $f(x)$ at point $x_0$:

$$g(\mathbf{x}) = f(\mathbf{x}_0) + \Re \left\{ \left( \nabla f(\mathbf{x}_0) \right)^{\mathrm{H}} (\mathbf{x} - \mathbf{x}_0) \right\} + \tfrac{1}{2} (\mathbf{x} - \mathbf{x}_0)^{\mathrm{H}} \mathbf{G} (\mathbf{x} - \mathbf{x}_0) .$$

- Objective after 1st majorization in quadratic form:

$$\left( \mathrm{vec}^{\mathrm{H}}(\tilde{\mathbf{Y}}) \right)^{\mathrm{H}} \left( \tilde{\mathbf{\Phi}} - \tfrac{\lambda_{\tilde{\Phi}}}{2} \mathbf{I}_{M^2P^2} \right) \mathrm{vec}(\tilde{\mathbf{Y}}^{(k)}).$$

- Objective after 2nd majorization in quadratic form:

$$\mathbf{y}^{\mathrm{H}} \left( \mathbf{B}^{(k)} - \tfrac{\lambda_{\tilde{\Phi}}}{2} \mathbf{y}^{(k)} (\mathbf{y}^{(k)})^{\mathrm{H}} \right) \mathbf{y}$$

$$\mathbf{B}^{(k)} \triangleq \begin{bmatrix} \mathbf{B}_{11}^{(k)} & \ldots & \mathbf{B}_{1M}^{(k)} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_{M1}^{(k)} & \ldots & \mathbf{B}_{MM}^{(k)} \end{bmatrix}; \qquad \boldsymbol{\rho}_{mm'}^{(k)} \triangleq \begin{cases} \gamma_p^2 \mathbf{1}_{P-p}^{\mathrm{T}} \mathcal{U}_p\left(\mathbf{Z}_{mm'}^{(k)}\right), & p \in \mathbf{\Omega} \\ 0, & p \in \bar{\mathbf{\Omega}} \end{cases}$$

$$\mathbf{B}_{mm'}^{(k)} = 2P\mathcal{T}\left( \boldsymbol{\rho}_{mm'}^{(k)}, \boldsymbol{\eta}_{mm'}^{(k)} \right); \qquad \boldsymbol{\eta}_{mm'}^{(k)} \triangleq \begin{cases} \gamma_p^2 \mathbf{1}_{P-p}^{\mathrm{T}} \mathcal{D}_p\left(\mathbf{Z}_{mm'}^{(k)}\right), & p \in \mathbf{\Omega} \\ 0, & p \in \bar{\mathbf{\Omega}} \end{cases}$$

$$\mathbf{Z}_{mm'}^{(k)} \triangleq \mathbf{y}_m^{(k)} (\mathbf{y}_{m'}^{(k)})^{\mathrm{H}};$$

$\mathbf{\Omega} \triangleq \{0\} \cup \{p | \gamma_p \neq 0, p > 0\}$; $\bar{\mathbf{\Omega}} \triangleq \{p | \gamma_p = 0, p > 0\}$; $\mathcal{U}_p(\cdot), \mathcal{D}_p(\cdot)$: $p$th off-diagonal vector element in the upper/lower triangular part of a matrix; $\mathcal{T}(\cdot,\cdot)$: construction of Hermitian matrix from vector(s).

- Final optimization problem:

$$\min_{\mathbf{y}} \ \mathbf{y}^{\mathrm{H}} \left( \mathbf{B}^{(k)} - \tfrac{\tau^{(k)} + MP\lambda_{\tilde{\Phi}}}{2} \mathbf{I}_{MP} \right) \mathbf{y}^{(k)}$$
$$\text{s.t.} \ \ |\mathbf{y}(p')| = 1, \ p' = 1, \ldots, MP.$$

- Closed-form solution:

$$\mathbf{y}(p') = \exp\{ j \cdot \arg(\mathbf{z}^{(k)}(p')) \}, \ p' = 1, \ldots, MP$$

with $\mathbf{z}^{(k)} \triangleq \left( (\tau^{(k)} + MP\lambda_{\tilde{\Phi}}) \mathbf{I}_{MP}/2 - \mathbf{B}^{(k)} \right) \mathbf{y}^{(k)}$.

## Fast WISL Minimization-Based Algorithm

- Algorithm

**Algorithm 1** Fast WISL Minimization-Based Algorithm

$k \leftarrow 0$, $\mathbf{y} \leftarrow$ unimodular sequence with random phases.
$\lambda_{\tilde{\Phi}} \triangleq 2MP\lambda_{\max}^2(\mathbf{\Gamma})$
**repeat**
  **procedure** WISLMAMI$(\mathbf{y}^{(k)})$
    Calculate $\boldsymbol{\rho}_{mm'}^{(k)}$ and $\boldsymbol{\eta}_{mm'}^{(k)}$, $m = 1, \ldots, M$; $m' = m, \ldots, M$.
    Construct $\mathbf{B}^{(k)}$ through $\mathbf{B}_{mm'}^{(k)} = \left(\mathbf{B}_{m'm}^{(k)}\right)^{\mathrm{H}} = 2P\mathcal{T}\left(\boldsymbol{\rho}_{mm'}^{(k)}, \boldsymbol{\eta}_{mm'}^{(k)}\right)$
    $\tau^{(k)} = \left\| \mathbf{B}^{(k)} - \frac{1}{2}\lambda_{\tilde{\Phi}} \mathbf{y}^{(k)} (\mathbf{y}^{(k)})^{\mathrm{H}} \right\|$
    $\mathbf{z}^{(k)} = \left( \frac{1}{2}(\tau^{(k)} + MP\lambda_{\tilde{\Phi}}) \mathbf{I}_{MP} - \mathbf{B}^{(k)} \right) \mathbf{y}^{(k)}$
    $\mathbf{y}^{(k+1)}(p') = e^{j \cdot \arg\left(\mathbf{z}^{(k)}(p')\right)}$, $p' = 1, \ldots, MP$
    $k \leftarrow k + 1$
  **end procedure**
**until** convergence

- Computational complexity:
  - Assume $\mathbf{\Omega}$ consists of $N_P$ ($0 < N_P \leq P$) elements.
  - Both $\rho_{mm'}^{(k)}$ and $\boldsymbol{\eta}_{mm'}^{(k)}$ need at most $N_P P$ operations if $\mathbf{Z}_{mm'}^{(k)}$ is given.
  - Calculation of $\mathbf{Z}_{mm'}^{(k)}$ costs $P^2$ operations.
  - Calculations are repeated only for subscripts $m = 1, \ldots, M$ and $m' = m, \ldots, M$.
  - Calculation of the vector $\mathbf{z}^{(k)}$ needs $M^2P^2$ operations.
  - Total number of operations is upper bounded by $((3M^2 - M)P^2 + (M^2 - M)N_P P)/2$, i.e., at most of order $\mathcal{O}((M-1)MP^2)$ suitable for large-scale optimization.

## Simulation Results (Convergence speed and Correlation properties)

- WISL performance comparisons of the algorithms tested versus code length for $M = 2$ waveforms for stopping criterion (i):

| | $P=32, M=2$ | | | | $P=128, M=2$ | | | | $P=512, M=2$ | | | | $P=1024, M=2$ | | | | $P=2048, M=2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Min.[a] | Ave.[b] | Time | Iter.[c] | Min. | Ave. | Time | Iter. | Min. | Ave. | Time | Iter. | Min. | Ave. | Time | Iter. | Min. | Ave. | Time | Iter. |
| WeCAN | 29.08 | 29.84 | 1.47 | 314 | 19.84 | 19.83 | 5.82 | 200 | 19.38 | 20.44 | 17.65 | 67 | 20.10 | 20.25 | 90.82 | 97 | 19.65 | 20.41 | 313.78 | 97 |
| WISLSong | 27.80 | 28.65 | 0.57 | 238 | -34.88 | -27.18 | 0.78 | 41 | -47.12 | -21.17 | 7.82 | 21 | -40.18 | -35.68 | 41.35 | 15 | -48.17 | -28.67 | 102.62 | 9 |
| WISNew | 27.75 | 28.44 | 1.05 | 94 | -65.74 | -31.76 | 0.37 | 26 | -80.31 | -50.37 | 2.05 | 7 | -85.54 | -73.83 | 6.15 | 6 | -78.85 | -62.46 | 20.45 | 5 |

[a] Min.: Obtained minimum WISL value (in dB). [b] Ave.: Obtained average WISL value (in dB). [c] Iter.: Number of conducted iterations.

- WISL performance comparisons of the algorithms tested versus code length for $M = 2$ waveforms for stopping criterion (ii):

| | $P=32, M=2$ | | | | $P=128, M=2$ | | | | $P=512, M=2$ | | | | $P=1024, M=2$ | | | | $P=2048, M=2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Min.[a] | Ave.[b] | Time | Iter.[c] | Min. | Ave. | Time | Iter. | Min. | Ave. | Time | Iter. | Min. | Ave. | Time | Iter. | Min. | Ave. | Time | Iter. |
| WeCAN | 28.20 | 28.92 | 1.58 | 374 | 18.70 | 19.61 | 7.59 | 267 | 20.72 | 20.84 | 18.47 | 84 | 20.12 | 20.17 | 56.86 | 71 | 21.24 | 21.32 | 176.46 | 66 |
| WISLSong | 27.48 | 28.59 | 0.84 | 298 | -40.31 | -39.38 | 0.80 | 51 | -47.63 | -46.42 | 8.55 | 14 | -42.83 | -39.65 | 33.24 | 20 | -49.44 | -48.81 | 75.34 | 16 |
| WISLNew | 27.67 | 28.25 | 0.30 | 149 | -49.91 | -37.68 | 0.20 | 24 | -59.52 | -55.09 | 1.22 | 7 | -79.34 | -74.87 | 4.15 | 6 | -72.53 | -61.87 | 13.79 | 5 |

[a] Min.: Obtained minimum WISL value (in dB). [b] Ave.: Obtained average WISL value (in dB). [c] Iter.: Number of conducted iterations.

- Auto- and cross-correlations of $M = 2$ designed waveforms with code length $P = 32$:

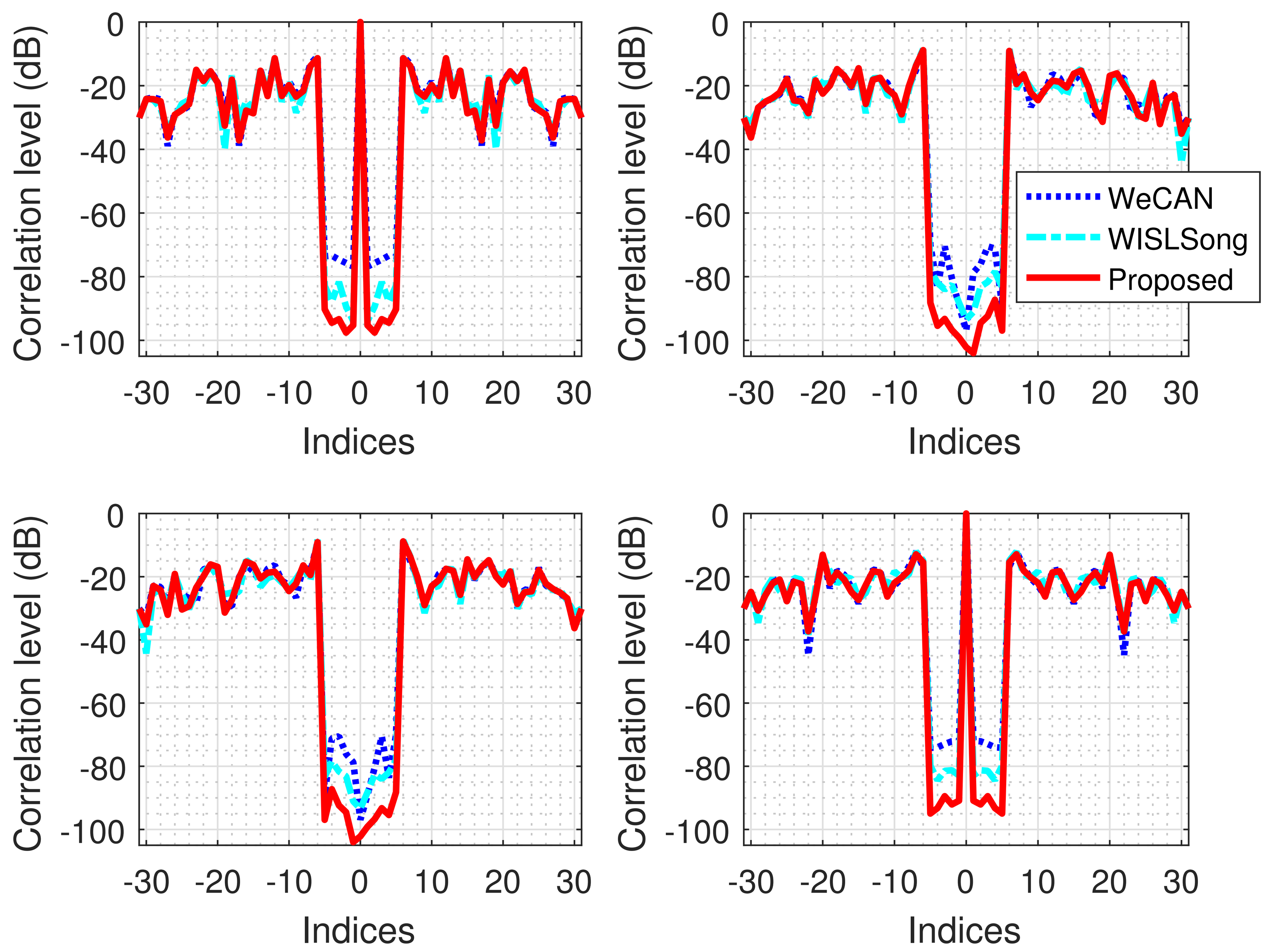


Figure 1: Evaluation of correlation properties.

## References


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