SUPER-RESOLUTION RESULTS FOR A 1D INVERSE SCATTERING PROBLEM

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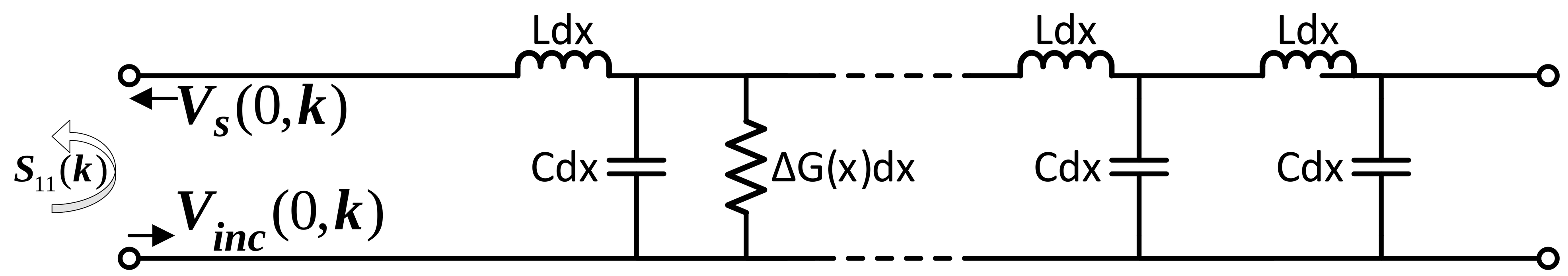
Motivation

- Distributed transmission line model



- Detecting faults in electrical cables and water pipelines, e.g. Blockages and Leakages.
- Conventional reconstruction is limited in resolution since the measurements are often restricted to low frequencies.
- Previous work [2] has proven that the location of discrete points in 1D could be precisely obtained if they are separated by at least $2/f_c$ given Fourier samples could be obtained up to f_c.

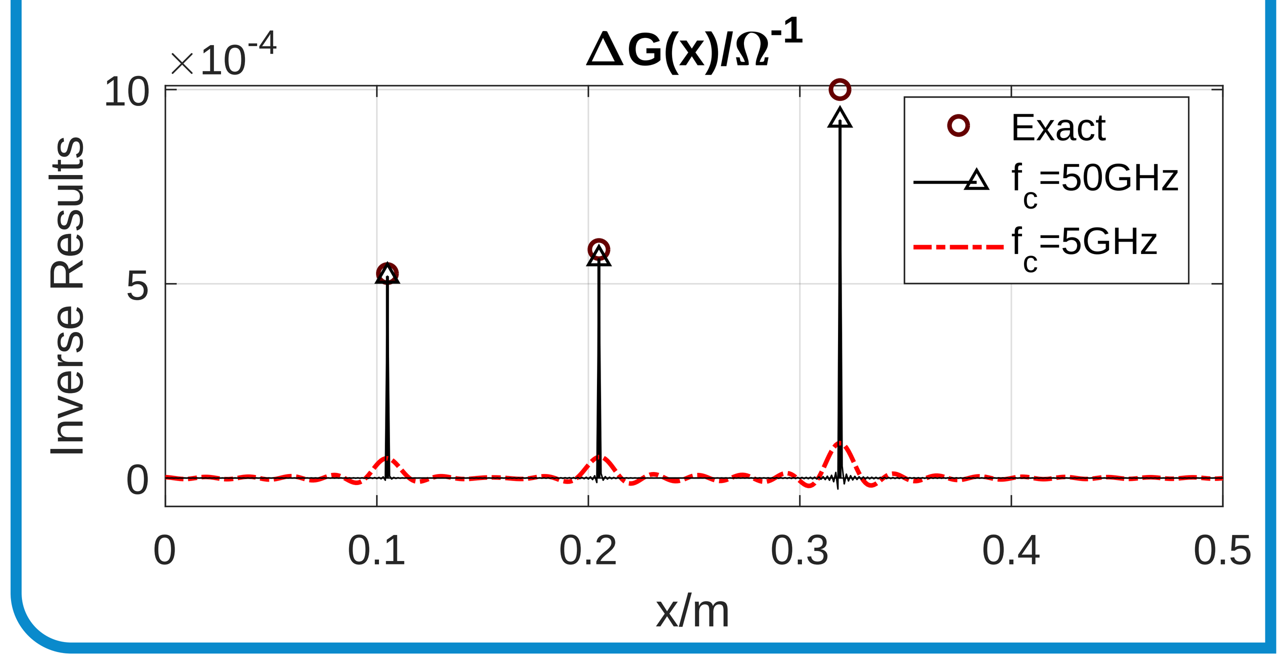
Model

- Point faults $\Delta G(x)$ model the point shunt conductance (partial short) in the transmission lines or the leakage in water pipelines.
- By using the Born approximation [1], the fault $\Delta G(x)$ are related to scattered parameter through a Fourier Transform:

$$\mathcal{F}^{-1}[S_{11}(k)](2x) = -\frac{1}{2}\Delta G(x) Z_0, \quad (1)$$

where the $S_{11}(k)$ is often limited in $|f| \leq f_c$.

- Conventional reconstruction results of point faults $\Delta G(x)$ with different f_c. 3 point faults ($\Delta G(0.105\text{m}) = 1/1900\Omega$, $\Delta G(0.205\text{m}) = 1/1700\Omega$ and $\Delta G(0.319\text{m}) = 1/1000\Omega$) are included.



Super-Resolution Scheme

- The inverse profile can be written as a weighted superposition of spikes

$$P = \mathcal{F}^{-1}[S_{11}(f)](t) = \sum_{j=1}^{s} a_j \delta_{t_j}. \quad (2)$$

- The band limited measurement S_{11} is related to the profile P as $S_{11} = \mathcal{F}_B P$ [2].
- To reconstruct P exactly from the band limited measurement S_{11}, the following ℓ_1 minimization objective has been proposed [2]:

$$\min_{\tilde{P}} \|\tilde{P}\|_{\ell_1} \quad \text{s.t.} \quad S_{11} = \mathcal{F}_B \tilde{P}. \quad (3)$$

Restricted Isometry Property

Definition 1. For each integer $s = 1, 2, ...,$ define the isometry constant δ_s of a linear map A as the smallest number such that

$$(1-\delta_s)\|x\|_{\ell_2}^2 \leq \|Ax\|_{\ell_2}^2 \leq (1+\delta_s)\|x\|_{\ell_2}^2$$

holds for such s-sparse signal x. x is said to be s-sparse if it has at most s nonzero entries.

Remark. *A necessary and sufficient condition for well-posedness of Eq. (3) with s-sparse P requires that $\hat{\mathcal{F}}_B$ satisfies the Restricted Isometry Property with $\delta_{2s} < 1$, where $\hat{\mathcal{F}}_B$ is $\mathcal{F}_B$ with its columns normalized.*

Conditional Well-posedness

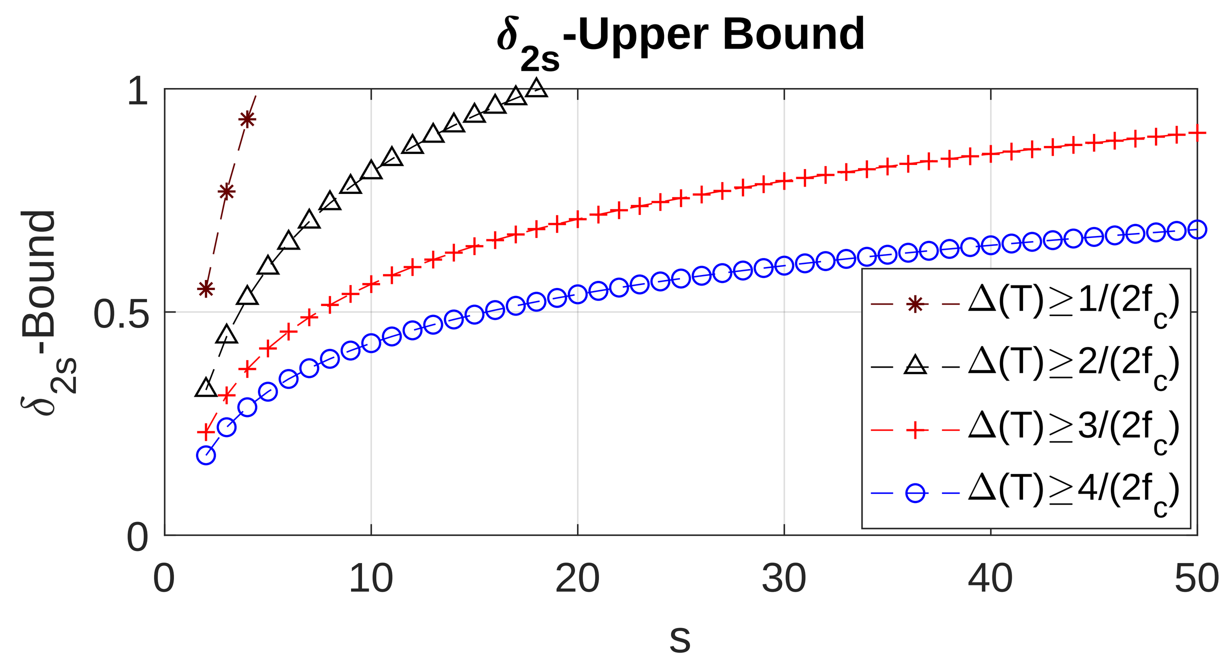
- The cross-correlations of any two columns, i.e. $\hat{\mathcal{F}}_{t_i}$ corresponds to t_i and $\hat{\mathcal{F}}_{t_j}$ corresponds to t_j, in $\hat{\mathcal{F}}_B$ can be written as

$$\frac{\left|\langle \hat{\mathcal{F}}_{t_i}, \hat{\mathcal{F}}_{t_j}\rangle\right|}{\|\hat{\mathcal{F}}_{t_i}\|_{\ell_2}\|\hat{\mathcal{F}}_{t_j}\|_{\ell_2}} = \left|\frac{\sin\left[\pi(2f_c+1)|t_i - t_j|\right]}{(2f_c+1)\sin(\pi|t_i - t_j|)}\right|.$$

- The cross-correlation of two consecutive columns in $\hat{\mathcal{F}}_B$ can be seen to be close to 1. Meanwhile, the cross-correlation of two columns in $\hat{\mathcal{F}}_B$ decreases if their separation increases.
- By restricting the minimal separation of discrete faults as $\Delta(T) \geq \frac{n}{2f_c}$ $(n \in \mathbb{N})$, the upper bound of the isometry constant will be

$$\delta_{2s} \leq \frac{\sum_{i \in E_{2s-1}}^{j \in E_{2s}\setminus E_i} c(t_i, t_j)\left[|p_{t_i}|^2 + |p_{t_j}|^2\right]}{\sum_{i=1}^{2s}|p_{t_i}|^2} \quad (4)$$

$$\leq \sum_{d=1}^{s-1} \frac{2}{(nd+0.5)\,\pi} + \frac{1}{(ns+0.5)\,\pi}.$$



Definition 2. Let $T = \{t_j\}$ be the support of P in Eq. (2), then the minimum separation of spikes in P is defined as

$$\Delta(T) = \min_{t,t' \in T, t \neq t'} |t - t'|_{\Delta},$$

where $|t - t'|_{\Delta}$ is the wrap-around distance, e.g. for $t \in [0,1]$, $t = 0$ and $t' = \frac{3}{4}$, $|t - t'|_{\Delta} = \frac{1}{4}$.

Results and Simulation

Theorem 1. *If $s \leq 4$, there exists a constant $\delta_{2s} < 1$ such that*

$$(1-\delta_{2s})\|P\|_{\ell_2}^2 \leq \|\hat{\mathcal{F}}_B P\|_{\ell_2}^2 \leq (1+\delta_{2s})\|P\|_{\ell_2}^2 \quad (5)$$

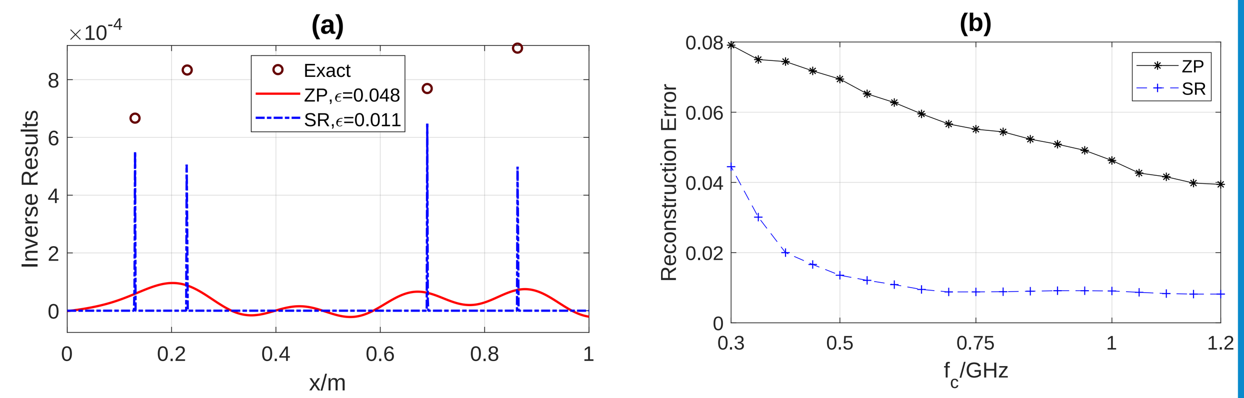
holds for any 2s-sparse signal P with its support satisfying $\Delta(T) \geq \frac{1}{2f_c}$.

Theorem 2. *If $5 \leq s \leq 18$, there exists a constant $\delta_{2s} < 1$ such that*

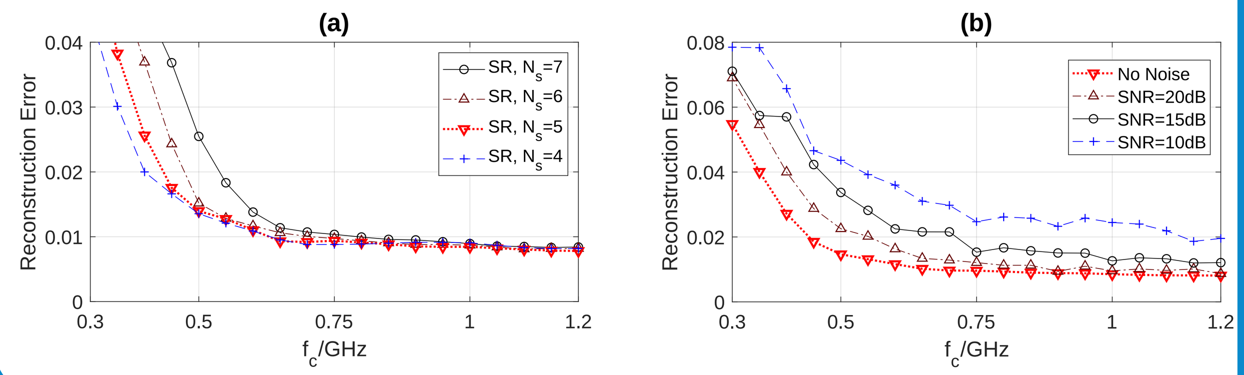
$$(1-\delta_{2s})\|P\|_{\ell_2}^2 \leq \|\hat{\mathcal{F}}_B P\|_{\ell_2}^2 \leq (1+\delta_{2s})\|P\|_{\ell_2}^2$$

holds for any 2s-sparse signal P with its support satisfying $\Delta(T) \geq \frac{1}{f_c}$.

- Reconstruction where $N_s = 4$. "ZP" is zero-padding and "SR" is super-resolution. (a) An example of the reconstruction where $f_c = 0.5$ GHz and (b) Reconstruction error (NRMSD) with respect to f_c when there is no noise.



- Reconstruction error with respect to f_c. (a) $N_s = \{7, 6, 5, 4\}$ with no noise and (b) $N_s = 5$ for various SNR levels.



Conclusions

- Super-resolution can be used to precisely locate faults when only limited bandwidth is available.

  Specifically, up to 4 discrete faults can be super-resolved if they are separated by at least $1/2f_c$ given Fourier samples could be obtained up to f_c.

- Simulations demonstrate the validity of the approach even when noise is included.

Reference

[1] Liwen Jing, Wenjie Wang, Zhao Li, and Ross Murch, "Detecting impedance and shunt conductance faults in lossy transmission lines," *IEEE Transactions on Antennas and Propagation*, 2018.

[2] Emmanuel J Candès and Carlos Fernandez-Granda, "Towards a mathematical theory of super-resolution," *Communications on Pure and Applied Mathematics*, vol. 67, no. 6, pp. 906–956, 2014.