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A Novel Approximate Lloyd-Max Quantizer and its Analysis

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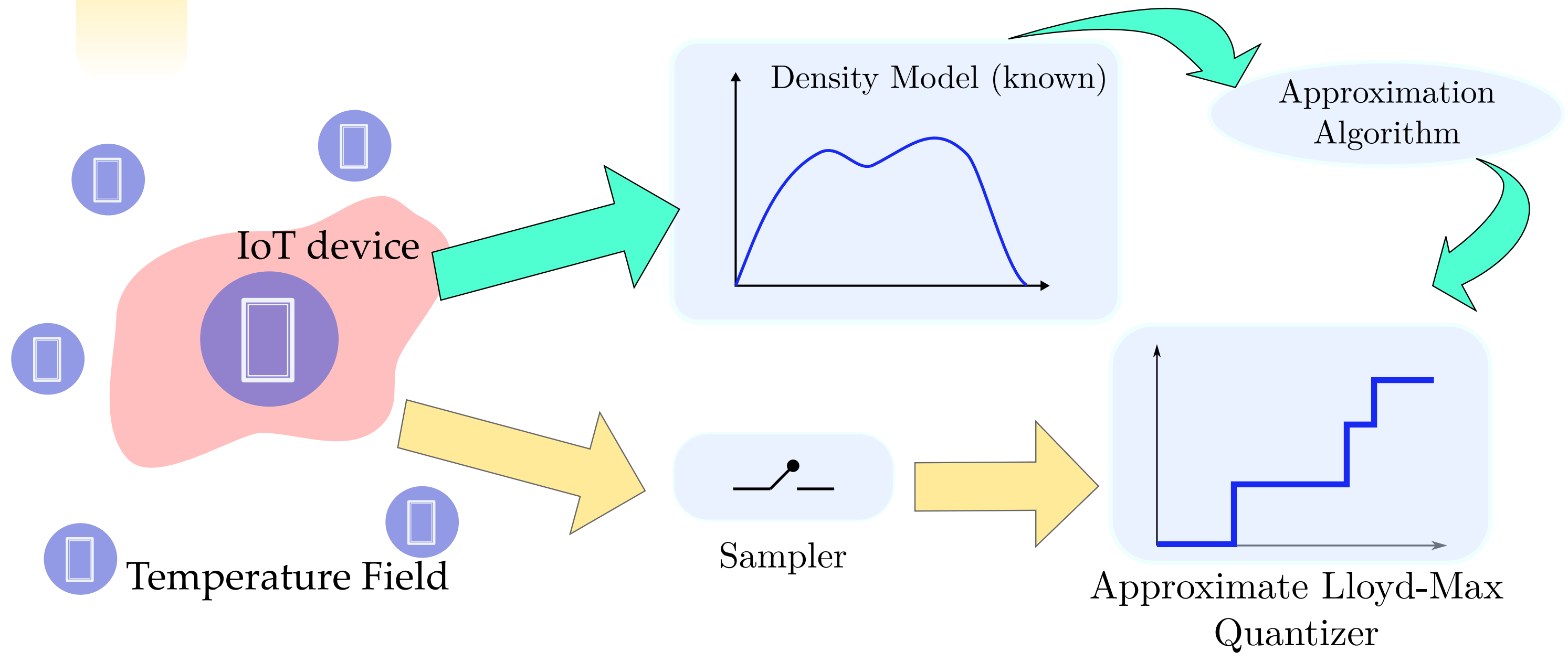
Introduction

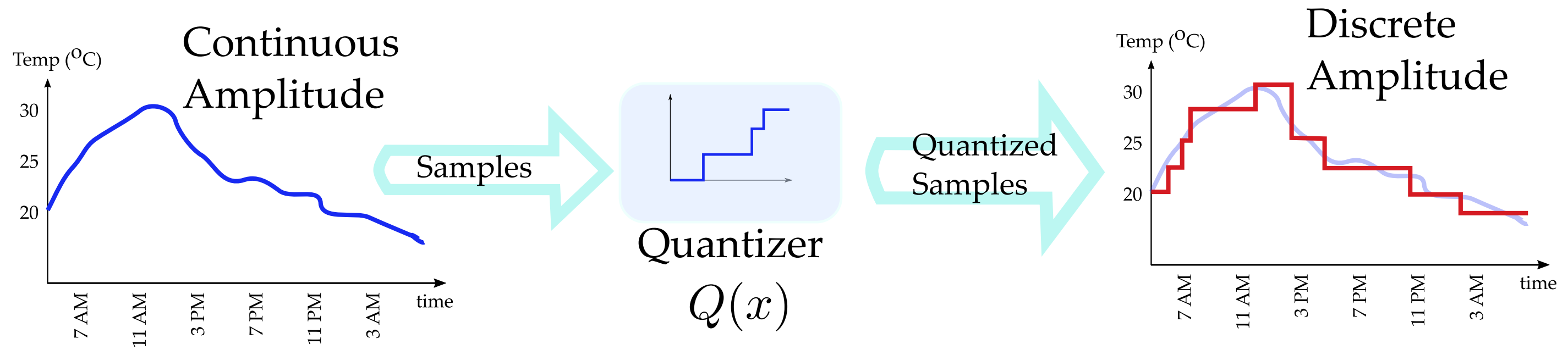
- Modern signal monitoring applications will rely on distributed IoT sensors
- Sensors have associated energy and computaional constraints
- Quantization of monitored signals becomes essential for efficient signal representation
- An **Approximate Lloyd-Max (ALM)** quantizer is proposed to speedup the quantizer design

System Model

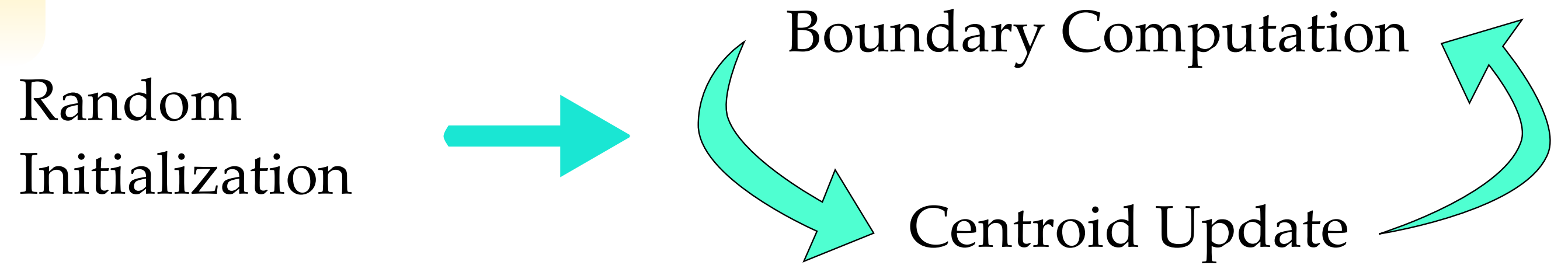


Quantization at a glance:

Find the Mean Squared Error (MSE) optimal quantizer



The classical Lloyd-Max [Lloyd'1982,Max'1960]

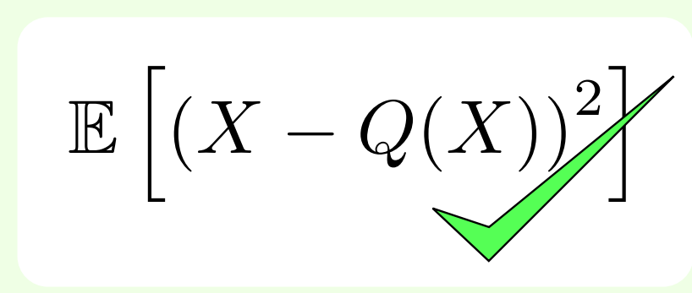


$$q_i = \frac{\int_{b_l}^{b_u} x f_X(x)\mathrm{d}x}{\int_{b_l}^{b_u} f_X(x)\mathrm{d}x}$$

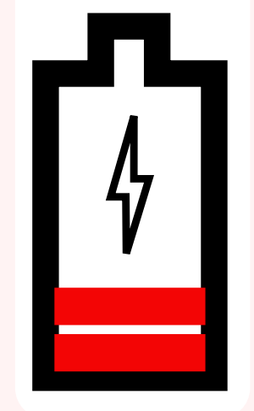
$f_X(x) \longrightarrow$ Known signal source density

b_u and $b_l \longrightarrow$ Upper /lower boundaries

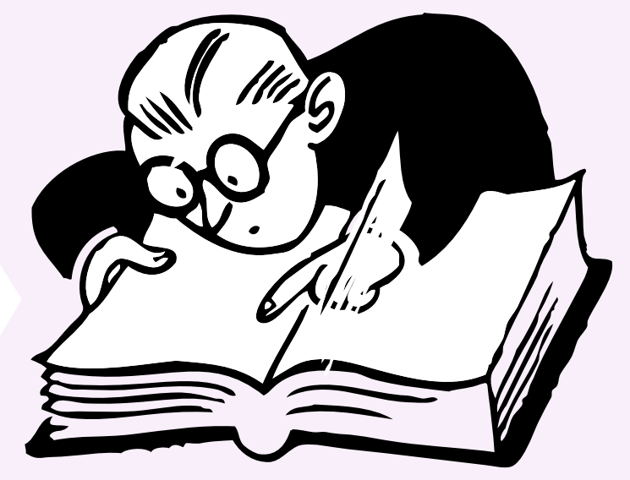
Features of Lloyd-Max Quantizer



MSE Optimal in many cases
Kieffer'1982



Computational energy required is high



Convergence analysis by state machine formulation
Wu'1992

Approximate Lloyd-Max

Highlights of ALM

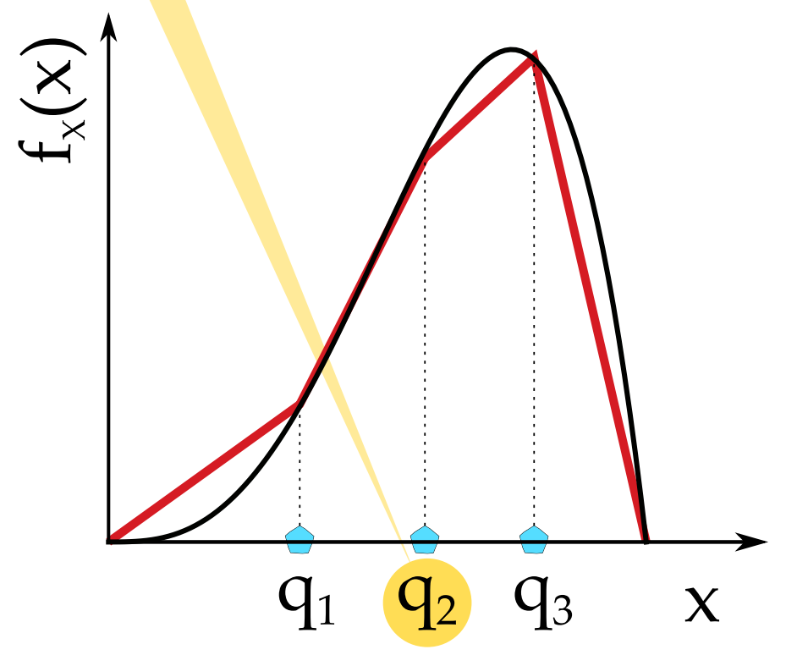
- *Bypasses* the centroid integration step
- Faster computation - *quadratic polynomial root solving*
- Analysis similar to *Perron-Frobenius* theory in Markov Chains [Gallager'2013]

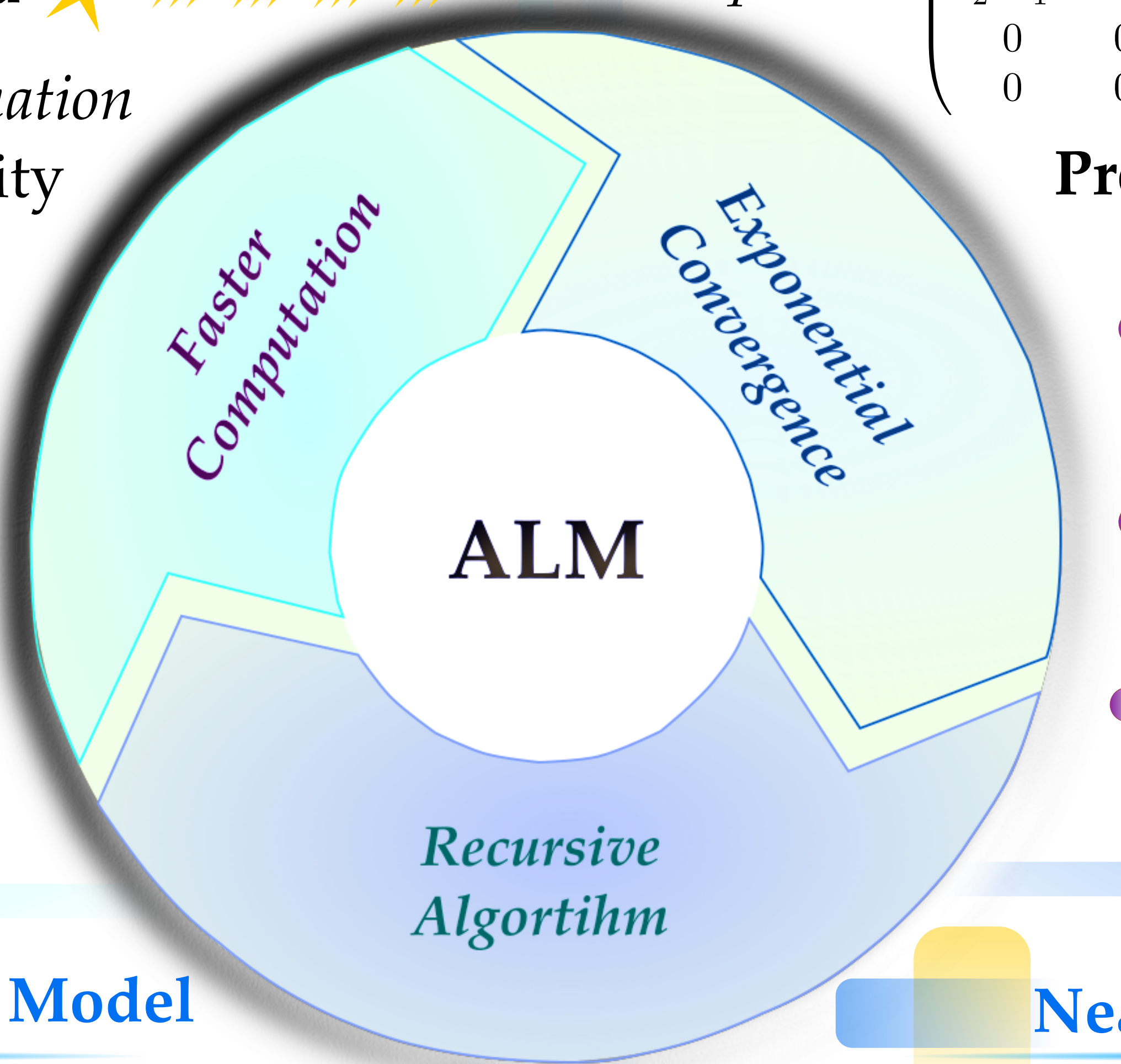
Key Idea

Piecewise-linear approximation of signal source density

$$0 = \int_{b_l}^{b_u} (q_2 - x) f_{\mathrm{app}}(x)\mathrm{d}x$$

$$f_{\mathrm{app}}(x) = m_2 x + c_2$$





Source and Quantizer Model

1. Quantizer cost function : **Mean Squared Error**
$$\mathcal{R}_Q = \mathbb{E}\left[(X - Q(X))^2\right]$$

2. Smoothness condition
$$|f'(x)| \le m \text{ for some } m \in [0, \infty)$$

3. Quantization Levels
$$q_1 \le q_2 \le \cdots \le q_K$$

Convergence of ALM

Linear update rule

$$q_k^{(i+1)} = \theta_k^{(i)} q_{k-1}^{(i)} + (1-\theta_k^{(i)}) q_{k+1}^{(i)}$$

Concurrent even-odd update

ALM updates in linear form

$$\vec{q}^{\,(i+1)} = \underbrace{P_{\mathrm{odd}}^{(i)} P_{\mathrm{even}}^{(i)}}_{P^{(i)}} \vec{q}^{\,(i)}$$

$\theta_k^{(i)}$ is a function of m_k and c_k

$$P^{(i)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \theta_1^{(i)} & 0 & \bar{\theta}_1^{(i)} & 0 & 0 \\ \theta_2^{(i)}\theta_1^{(i)} & 0 & \bar{\theta}_1^{(i)}\theta_2^{(i)} + \bar{\theta}_2^{(i)}\theta_3^{(i)} & 0 & \bar{\theta}_2^{(i)}\bar{\theta}_3^{(i)} \\ 0 & 0 & \theta_3^{(i)} & 0 & \bar{\theta}_3^{(i)} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

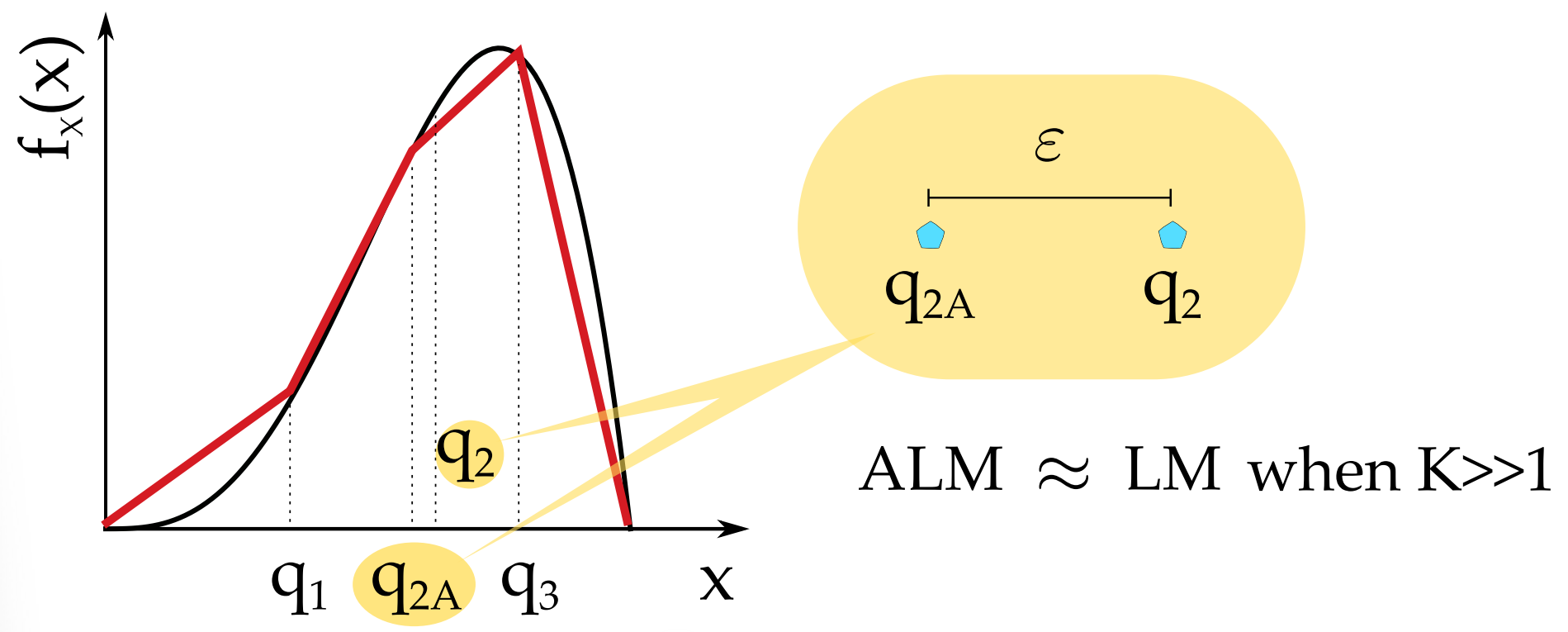
Properties of the update matrix

- *Row Stochasticity :*
$$\sum_m [P]_{lm} = 1$$
- *Updates bounded away :*
$$0 < \theta_k^{(i)} < 1$$
- *Even index columns are zero vectors*

Near-optimality

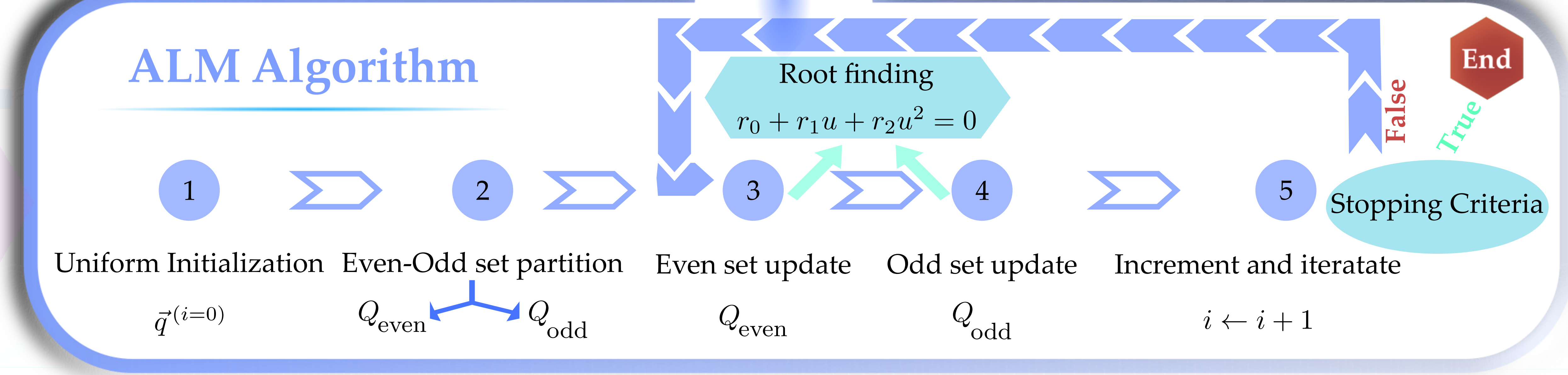
Theorem 1

$q_{2\mathrm{A}}$ converges to the true solution, q_2 as $K \to \infty$. That is, for all $\varepsilon > 0$ there exists a $K \ge K_0$ such that $|q_{2\mathrm{A}} - q_2| \le \varepsilon$. $\varepsilon = O(\frac{1}{K})$



ALM $\approx$ LM when K>>1

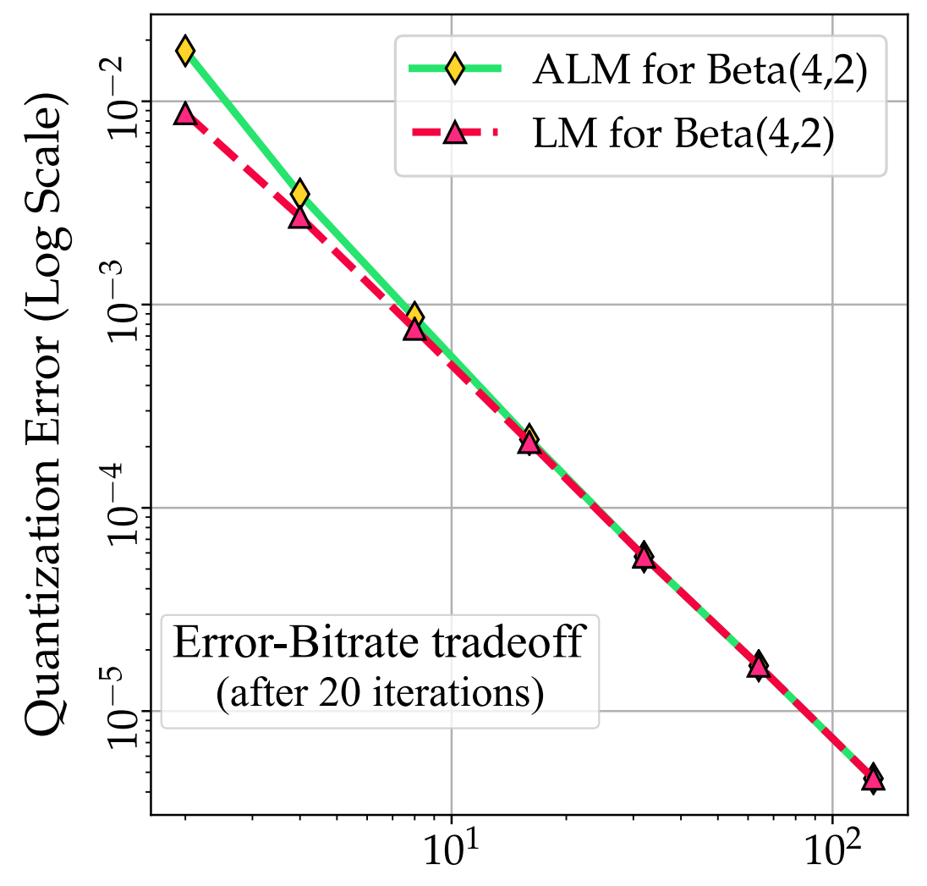
ALM Algorithm

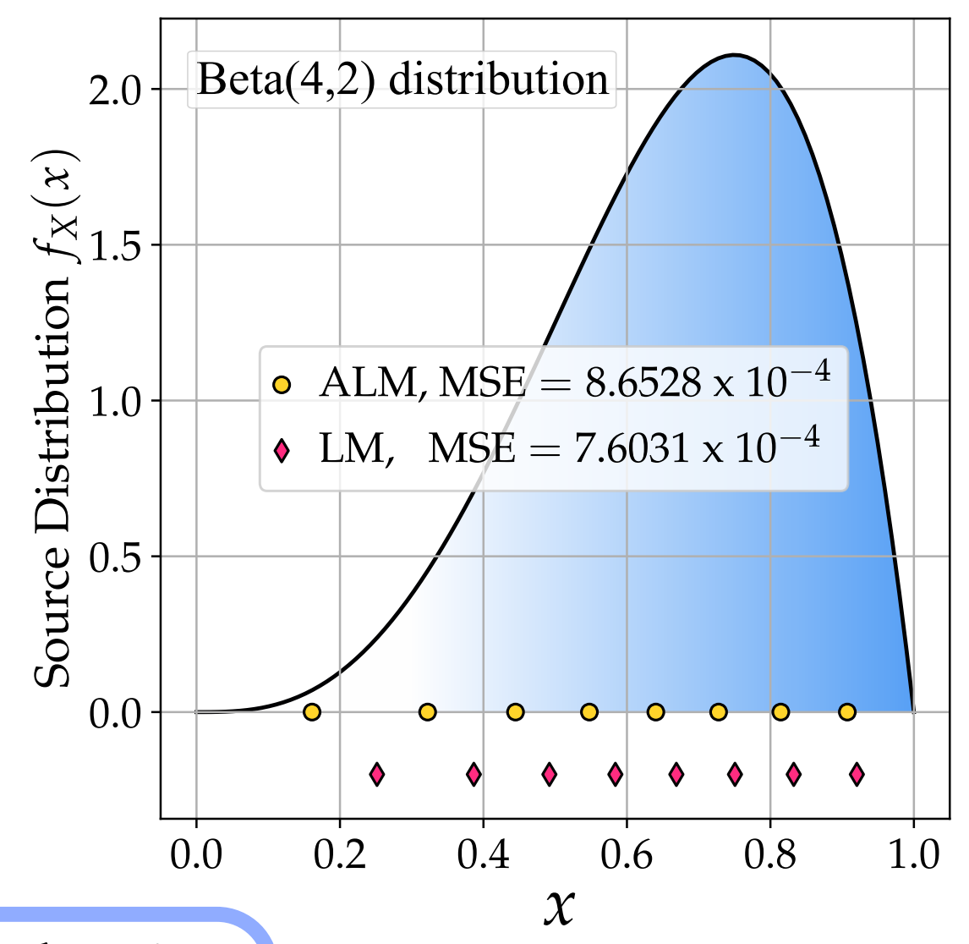


Root finding
$$r_0 + r_1 u + r_2 u^2 = 0$$

1. Uniform Initialization
$\vec{q}^{\,(i=0)}$
2. Even-Odd set partition
Q_{even} Q_{odd}
3. Even set update
Q_{even}
4. Odd set update
Q_{odd}
5. Increment and iteratate
$i \leftarrow i+1$

Simulation Results

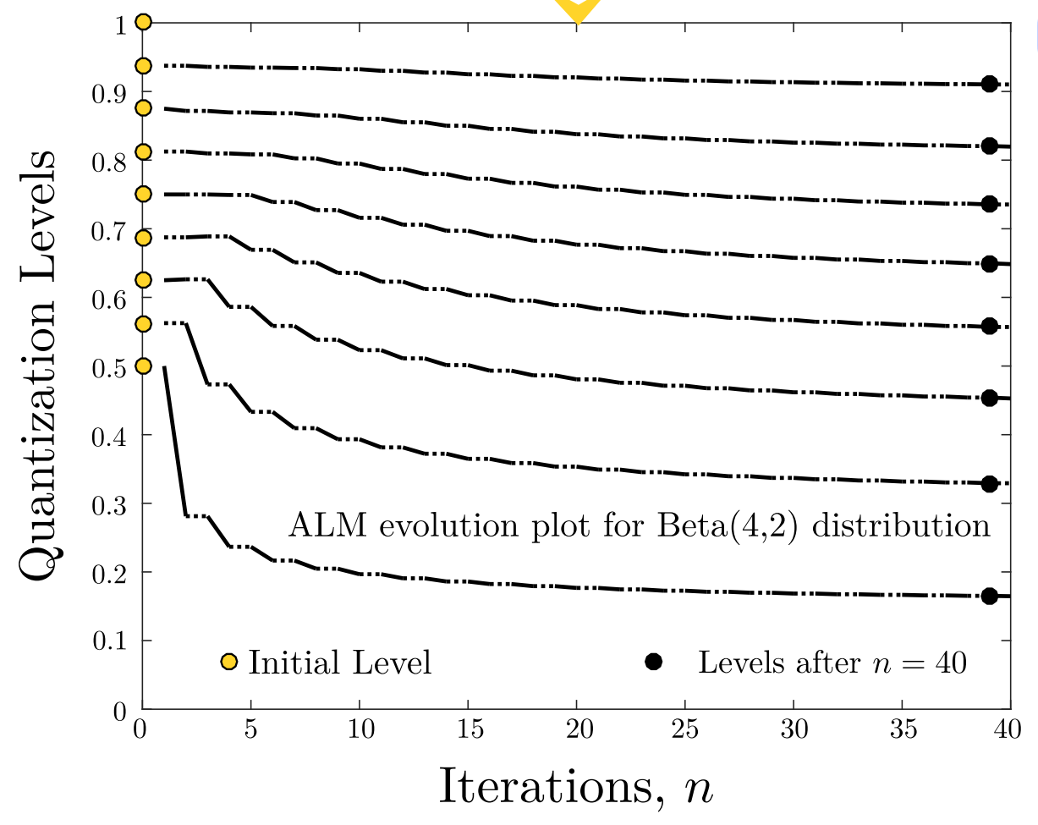




Tradeoff

Evolution

Accuracy



Mean squared error approx. matches for K > 8

3.4X speedup for ALM at K=8

Main convergence result

Theorem 2

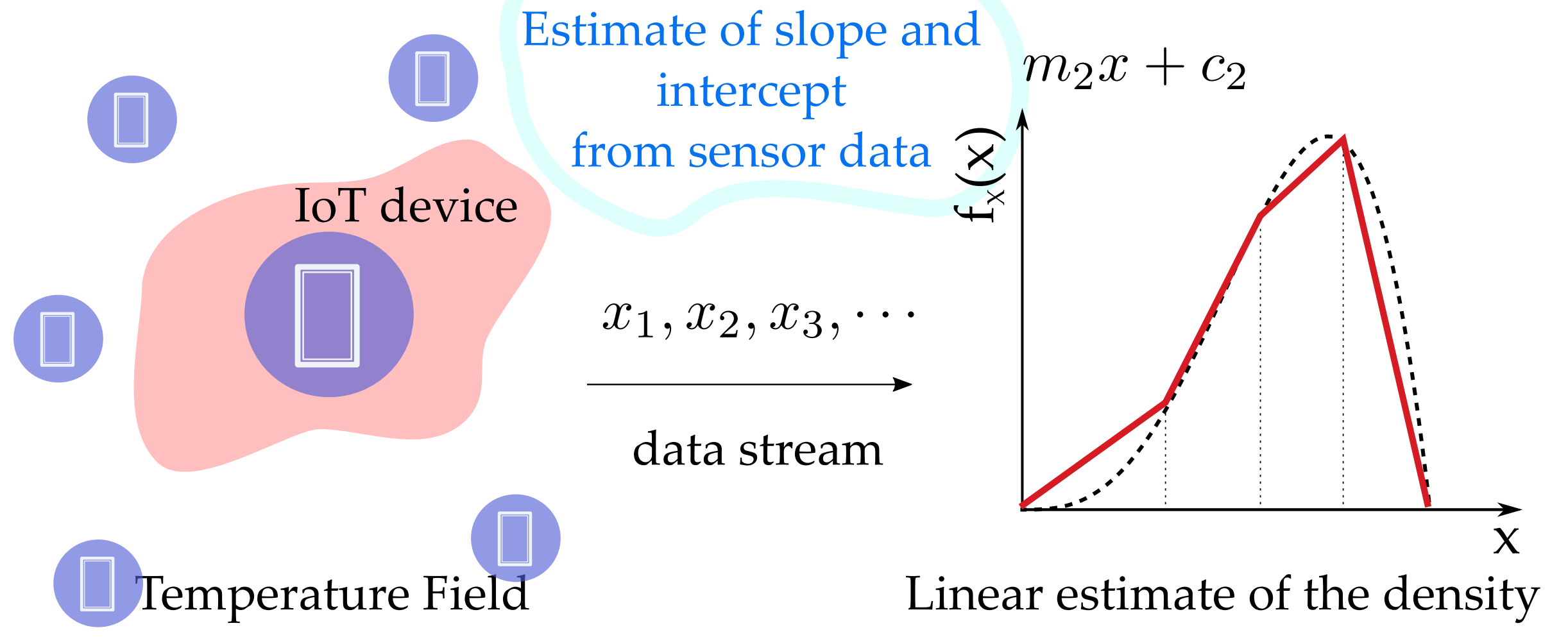
The ALM iterations converge to a quantization vector

$$\vec{q}^{\,*} = P^* \vec{q}^{\,(0)},$$

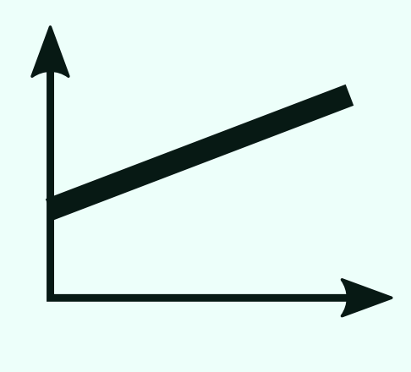
where $P^* = P^{(1)} P^{(2)} \ldots$ and $\vec{q}^{\,*}$ is independent of the initialization $\vec{q}^{\,(0)}$.

Learning ALM

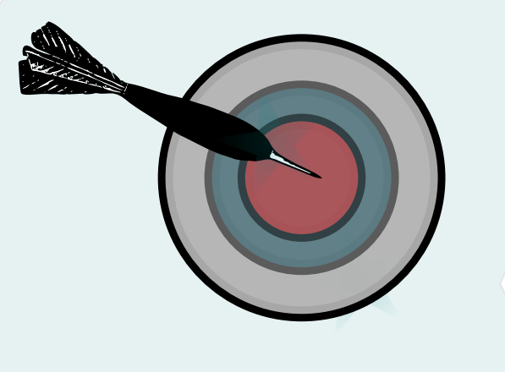
Piecewise linear approximation of the density function is learned from data samples



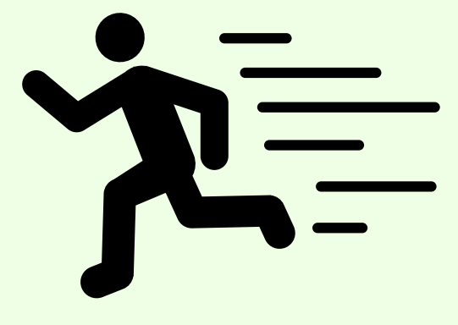
Summary of
Features of Approximate Lloyd-Max Quantizer



Nearly-linear updates



Near-optimal quantizer



Exponentially fast convergence