Distributed Tracking of Maneuvering Target: A Finite-Time Algorithm

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Problem Formulation

- Sensor network as an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ of order n
  - Stationary sensors located at positions, $\mathbf{s}_i \in \mathbb{R}^2$
- Target position $\boldsymbol{p}(t) \in \mathbb{R}^2$

$$\dot{\boldsymbol{p}}(t) = \mathbf{v}(t)$$

- Measurements are unit vectors $\boldsymbol{\varphi}_i(t)$

$$\boldsymbol{\varphi}_i(t) = \frac{\boldsymbol{p}(t) - \boldsymbol{s}_i}{\|\boldsymbol{p}(t) - \boldsymbol{s}_i\|_2}$$

- Define $\rho_i(t) = \|\boldsymbol{p}(t) - \boldsymbol{s}_i\|_2$ and $\boldsymbol{\varphi}_i(t) = \left[\cos(\theta_i(t)) \quad \sin(\theta_i(t))\right]^\top$

$$\rho_i(t)\boldsymbol{\varphi}_i(t) = \boldsymbol{p}(t) - \boldsymbol{s}_i$$

Proposition 1 *Let $\bar{\boldsymbol{\varphi}}_i(t) \in \mathcal{S}^1$ be an orthonormal vector obtained by rotating $\boldsymbol{\varphi}_i(t)$ by $\pi/2$ radians clockwise. Then*

$$\bar{\boldsymbol{\varphi}}_i(t) = \left[-\sin(\theta_i(t)) \quad \cos(\theta_i(t))\right]^\top$$

$$\boldsymbol{\varphi}_i(t)\boldsymbol{\varphi}_i^\top(t) + \bar{\boldsymbol{\varphi}}_i(t)\bar{\boldsymbol{\varphi}}_i^\top(t) = I_2.$$

- Measurements: $\bar{\boldsymbol{\varphi}}_i^\top(t)\boldsymbol{s}_i = \bar{\boldsymbol{\varphi}}_i^\top(t)\boldsymbol{p}(t)$
- Let

$$\mathbf{H}(t) = \begin{bmatrix} \mathbf{h}_1^\top(t) \\ \mathbf{h}_2^\top(t) \\ \vdots \\ \mathbf{h}_n^\top(t) \end{bmatrix}, \quad \mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

  where $\mathbf{h}_i^\top(t) = \bar{\boldsymbol{\varphi}}_i^\top(t)$ and $z_i(t) = \bar{\boldsymbol{\varphi}}_i^\top(t)\boldsymbol{s}_i$

- Measurements for the entire network

$$\mathbf{z}(t) = \mathbf{H}(t)\boldsymbol{p}(t).$$

Assumption 1 $\operatorname{rank}(\mathbf{H}(t)) = 2 < n$.

- Unique solution:

$$\boldsymbol{p}^*(t) = \left(\mathbf{H}^\top(t)\mathbf{H}(t)\right)^{-1}\mathbf{H}^\top(t)\mathbf{z}(t).$$

Goal: Estimate $\boldsymbol{p}^*(t)$ distributedly via local interactions

Distributed Algorithm

- In terms of local quantities $\boldsymbol{p}^*(t) =$

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{h}_i(t)\mathbf{h}_i^\top(t)\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{h}_i(t)z_i(t)\right)$$

- Let $\mathbf{P}_i(t) = \mathbf{h}_i(t)\mathbf{h}_i^\top(t)$ and $\mathbf{q}_i(t) = z_i(t)\mathbf{h}_i(t)$

$$\boldsymbol{p}^*(t) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{P}_i(t)\right)^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{q}_i(t)$$

- Construct a vector $\boldsymbol{\phi}_i(t) \in \mathbb{R}^6$

$$\boldsymbol{\phi}_i(t) = \begin{bmatrix} \operatorname{vec}(\mathbf{P}_i(t)) \\ \mathbf{q}_i(t) \end{bmatrix}$$

- Time-varying average

$$\bar{\boldsymbol{\phi}}(t) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\phi}_i(t) = \frac{1}{n}\left(\mathbf{1}_n^\top \otimes I_6\right)\boldsymbol{\phi}(t)$$

Assumption 2 *There exists a positive constant $\gamma > 0$ such that $\forall\, i \in \mathcal{I}$*

$$\sup_{t\in[t_0,\infty)}\|\dot{\boldsymbol{\phi}}_i(t)\|_\infty \leq \gamma < \infty$$

Assumption 3 *The interaction topology of n networked sensors is given as an unweighted connected undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$.*

Lemma 1 *For any strongly connected, weight-balanced graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ of order n, the graph Laplacian $\mathcal{L}$ is a positive semi-definite matrix with a single eigenvalue at 0 corresponding to both the left and right eigenvectors $\mathbf{1}_n^\top$ and $\mathbf{1}_n$, respectively.*

Lemma 2 *Let $M \triangleq \left(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top\right)$. For any connected undirected network $\mathcal{G}(\mathcal{V},\mathcal{E})$ of order n, the graph Laplacian $\mathcal{L}$ and the incidence matrix $\mathcal{B}$ satisfy*

$$M = \mathcal{L}(\mathcal{L})^+ = \mathcal{B}\mathcal{B}^\top\left(\mathcal{B}\mathcal{B}^\top\right)^+ = \mathcal{B}\left(\mathcal{B}^\top\mathcal{B}\right)^+\mathcal{B}^\top,$$

where $(\cdot)^+$ denotes the generalized inverse.

Dynamic average consensus (DAC)

- DAC algorithm

$$\dot{\boldsymbol{w}}_i(t) = -\beta\sum_{j=1}^{n}a_{ij}\operatorname{sgn}\left\{\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\right\}$$

$$\boldsymbol{x}_i(t) = \boldsymbol{w}_i(t) + \boldsymbol{\phi}_i(t)$$

- $\boldsymbol{w}_i(t) \in \mathbb{R}^6$ is the internal states
- $\boldsymbol{x}_i(t) \in \mathbb{R}^6$ is the estimate of $\bar{\boldsymbol{\phi}}(t)$
- In a compact form

$$\dot{\mathbf{w}}(t) = -\beta\left(\mathcal{B}\otimes I_6\right)\operatorname{sgn}\left\{\left(\mathcal{B}^\top \otimes I_6\right)\mathbf{x}(t)\right\}$$

$$\mathbf{x}(t) = \mathbf{w}(t) + \boldsymbol{\phi}(t),$$

- Define

$$\mathbf{w}(t) \in \mathbb{R}^{n6} \triangleq \left[\boldsymbol{w}_1^\top(t) \quad \dots \quad \boldsymbol{w}_n^\top(t)\right]^\top$$

$$\tilde{\mathbf{x}}(t) \triangleq \mathbf{x}(t) - \mathbf{1}_n \otimes \bar{\boldsymbol{\phi}}(t)$$

- Average-consensus error for the entire network (Lemma 2)

$$\tilde{\mathbf{x}}(t) = \mathbf{w}(t) + (M \otimes I_6)\,\boldsymbol{\phi}(t)$$

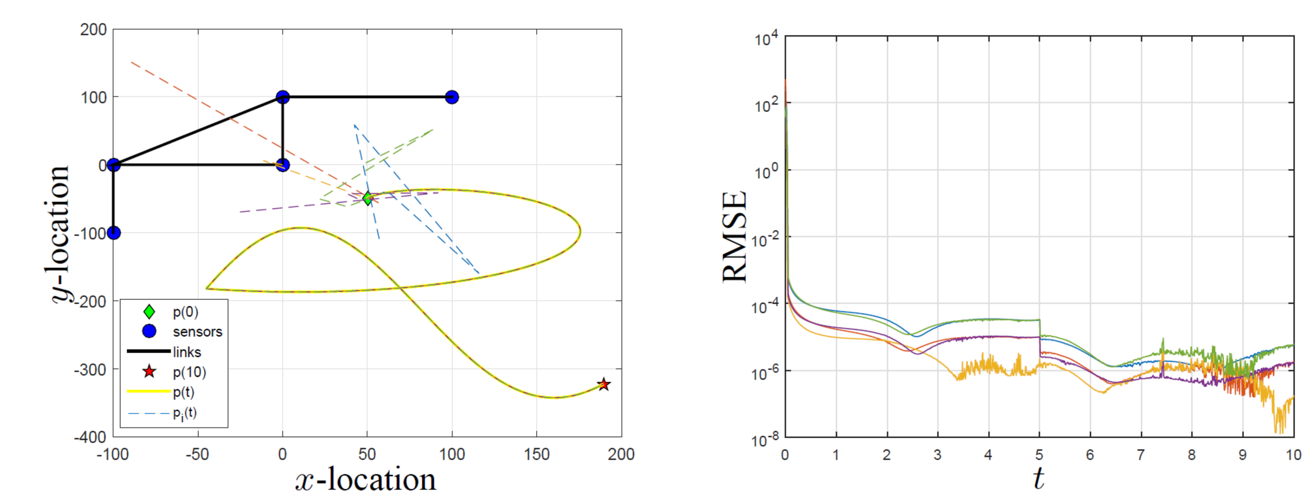
Theorem 1 *Given Assumptions 2 and 3, the robust dynamic average-consensus algorithm guarantees that the consensus error, $\tilde{\mathbf{x}}(t)$, is globally finite-time convergent, i.e., $\forall\,\tilde{\mathbf{x}}(t_0)$, we have $\tilde{\mathbf{x}}(t) = \mathbf{0}$ for all $t \geq t^*$, where $t^* = t_0 + \frac{\|\tilde{\mathbf{x}}(t_0)\|_2}{\lambda_2(L)}$, if $\mathbf{w}(t_0)$ is set to zero and β is selected such that $\beta \geq 1 + \gamma\frac{\sqrt{\hat{n}}}{\hat{\lambda}_2}$, where $\hat{n}$ and $\hat{\lambda}_2$ are positive constants such that $\hat{n} \geq n$ and $\hat{\lambda}_2 \leq \lambda_2(L)$, where $\lambda_2(L)$ is the algebraic connectivity of the network.*

Theorem 2 *Given Assumptions 1, 2, and 3, the proposed approach guarantees that the individual solutions $\boldsymbol{p}_i(t)$ converges to the optimal solution $\boldsymbol{p}^*(t)$ in finite time, i.e., for all $t \geq t^*$, $\boldsymbol{p}_i(t) = \boldsymbol{p}^*(t)$.*

Algorithm 1 Distributed tracking algorithm

Initialization : $\mathbf{w}(t_0) = \mathbf{0}_{6n}$
for $t \geq t_0$ **do**
  for $i = 1$ to n **do**
    Obtain: $z_i(t)$ & $\mathbf{h}_i^\top(t)$
    $\mathbf{P}_i(t) = \mathbf{h}_i(t)\mathbf{h}_i^\top(t)$
    $\mathbf{q}_i(t) = z_i(t)\mathbf{h}_i(t)$
    $\boldsymbol{\phi}_i(t) = \begin{bmatrix} \operatorname{vec}(\mathbf{P}_i(t)) \\ \mathbf{q}_i(t) \end{bmatrix}$
    $\boldsymbol{x}_i(t) = \boldsymbol{w}_i(t) + \boldsymbol{\phi}_i(t)$
    $\dot{\boldsymbol{w}}_i(t) = -\beta\sum_{j=1}^{n}a_{ij}\operatorname{sgn}\left\{\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\right\}$
    $\mathbf{P}_{\boldsymbol{x}_i}(t) \Leftarrow [\boldsymbol{x}_i(t)]_{1:4}$
    $\mathbf{q}_{\boldsymbol{x}_i}(t) \Leftarrow [\boldsymbol{x}_i(t)]_{5:6}$
    $\boldsymbol{p}_i(t) = \left(\mathbf{P}_{\boldsymbol{x}_i}(t)\right)^{-1}\mathbf{q}_{\boldsymbol{x}_i}(t)$
  end for
end for

Numerical Results



(a) Simulation scenario (b) Tracking error for all 5 nodes

Parameters: $\gamma = 10^2$, $\hat{n} = 5$, and $\hat{\lambda}_2 = 0.4$

Conclusion

- Distributed algorithm to track maneuvering targets from bearing measurements
- Built on the dynamic average consensus algorithm
- Can be easily extended to discrete-time scenarios
- Future research include extension to noisy scenarios and privacy preserving & event-triggered communication schemes