Sequential Monte Carlo sampling for correlated latent long-memory time-series

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Summary

- Study of hidden correlated time-series
  - long-memory, self-similarity and scale-invariance properties
  - operator fractional Gaussian processes
- Sequential Monte Carlo (SMC) for
  - sequential processing of data
  - non-Gaussian densities
  - nonlinear observations
- Filtering and forecasting

Motivation

In finance and econometrics

- Multi-scale and self-similar data
- Multivariate data with correlated variables

Operator fractional Gaussian process (OfGp)

- Multivariate Gaussian
- Self-similarity (Hurst parameter H)

Problem statement

We study the following state-space model

$$\begin{cases} u_{i,t} \sim fGp(H_i, \sigma_i^2), & i = 1, \cdots, d_u, \\ x_t = A u_t + \epsilon_t, & \epsilon_t \sim \mathcal{N}(0, C_\epsilon), \\ y_t = h(x_t, v_t). \end{cases}$$

$u_{i,t}$ is a fractional Gaussian process (fGp), with

$$\gamma_{u_i}(\tau) = \frac{\sigma_{u_i}^2}{2}\left[|\tau-1|^{2H_i} - 2|\tau|^{2H_i} + |\tau+1|^{2H_i}\right].$$

$x_t \in \mathbb{R}^{d_x \times 1}$ is OfGp embedded in noise.

Illustrative application

The stochastic volatility model with correlated trend

$$\begin{cases} u_{1,t} \sim fGp(H_1, \sigma_1^2), \\ u_{2,t} \sim fGp(H_2, \sigma_2^2), \\ \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} + \mathcal{N}(0, \sigma_\epsilon^2 I), \\ y_t = x_{1,t} + e^{x_{2,t}/2} v_t, \qquad v_t \sim \mathcal{N}(0,1). \end{cases}$$

Foundations of the proposed SMC method

1. **Sampling** from transition density
   - Hierarchical model

$$f(u_{t+1}, x_{t+1} | u_{1:t}, x_{1:t}) = f(u_{t+1}|u_{1:t}) f(x_{t+1}|u_{t+1})$$

2. **Weights** proportional to likelihood of observed data
3. Bayesian analysis to derive
   - fGp transition density with **known and unknown innovation variance**.
   - OfGp transition density with **known and unknown mixing parameters**.

Details of the proposed SMC method

Consider that at time instant t the discrete random measure is given by

$$\chi_t = \left\{\overline{u}_{1:t}^{(m_u)}, \overline{x}_{1:t}^{(m_u)}, w_t^{(m_u)}\right\}, \text{ where } m_u = 1, \cdots, M_u.$$

Upon reception of a new observation at time instant $t+1$:

1. Compute $\rho_{u_i}(\tau) = \frac{1}{2}\left[(\tau+1)^{2H_i} - 2\tau^{2H_i} + (\tau-1)^{2H_i}\right]$.
2. Propagate latent fGp samples
   - If σ_i^2 is known,

$$u_{i,t+1}^{(m_u)} \sim f(u_{i,t+1}^{(m_u)} | \overline{u}_{i,1:t}^{(m_u)}) = \mathcal{N}\left(\mu_{i,t+1}^{(m_u)}, \sigma_{i,t+1}^2\right), \qquad \begin{cases} \mu_{i,t+1}^{(m_u)} = c_{i,t} C_{i,t}^{-1} \overline{u}_{i,1:t}^{(m_u)}, \\ \sigma_{i,t+1}^2 = \sigma_i^2 \left(\rho_{u_i}(0) - c_{i,t} C_{i,t}^{-1} c_{i,t}^\top\right). \end{cases}$$

   - If σ_i^2 is unknown,

$$u_{i,t+1}^{(m_u)} \sim f(u_{i,t+1}^{(m_u)} | \overline{u}_{i,1:t}^{(m_u)}) = \mu_{i,t+1}^{(m_u)} + l_{i,t+1}^{(m_u)} \mathcal{T}(\nu_{t+1}), \qquad \begin{cases} \nu_{t+1} = \nu_0 + t, \\ \mu_{i,t+1}^{(m_u)} = c_{i,t} C_{i,t}^{-1} \overline{u}_{i,1:t}^{(m_u)}, \\ \sigma_{i,t}^{2(m_u)} = \frac{\nu_0 \sigma_0 + \overline{u}_{i,1:t}^{(m_u)} C_{i,t}^{-1} \left(\overline{u}_{i,1:t}^{(m_u)}\right)^\top}{\nu_t}, \\ l_{i,t+1}^{(m_u)^2} = \sigma_{i,t}^{2(m_u)} \left(\rho_{u_i}(0) - c_{i,t} C_{i,t}^{-1} c_{i,t}^\top\right). \end{cases}$$

3. Propagate M_x latent state particles
   - If the mixing parameters are known,

$$x_{t+1}^{(m_u,m_x)} \sim f(x_{t+1} | u_{t+1}^{(m_u)}) = \mathcal{N}(A u_{t+1}^{(m_u)}, C_\epsilon).$$

   - If the mixing parameters are unknown,

$$x_{t+1}^{(m_u,m_x)} \sim f(x_{t+1} | u_{t+1}^{(m_u)}) = \mathcal{T}(\nu_{t+1}, \mu_{t+1}^{(m_u)}, R_{t+1}^{(m_u)}), \qquad \begin{cases} \nu_{t+1} = t - d_x - d_u + 1, \\ \hat{A}_t^{(m_u)} = X_t^{(m_u)} (U_t^{(m_u)})^\top (U_t^{(m_u)} (U_t^{(m_u)})^\top)^{-1}, \\ \mu_{t+1}^{(m_u)} = \hat{A}_t^{(m_u)} u_{t+1}^{(m_u)}, \\ R_{t+1}^{(m_u)} = \frac{\left(X_t^{(m_u)} - \hat{A}_t^{(m_u)} U_t^{(m_u)}\right)\left(X_t^{(m_u)} - \hat{A}_t^{(m_u)} U_t^{(m_u)}\right)^\top}{\nu_{t+1}\left(1 - (u_{t+1}^{(m_u)})^\top (U_{t+1}^{(m_u)} (U_{t+1}^{(m_u)})^\top)^{-1} u_{t+1}^{(m_u)}\right)}. \end{cases}$$

4. Compute the non-normalized weights

$$\widetilde{w}_{t+1}^{(m_u,m_x)} \propto f(y_{t+1} | x_{t+1}^{(m_u,m_x)}),$$

   and normalize them to obtain a new random measure

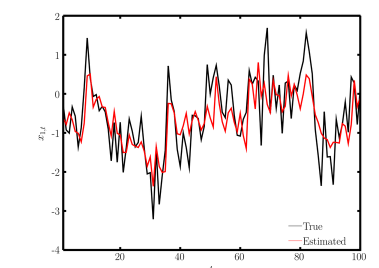
$$\chi_{t+1} = \left\{u_{1:t+1}^{(m_u)}, x_{1:t+1}^{(m_u,m_x)}, w_{t+1}^{(m_u,m_x)}\right\}.$$

5. Downsample from $M_u \times M_x$ to M_u

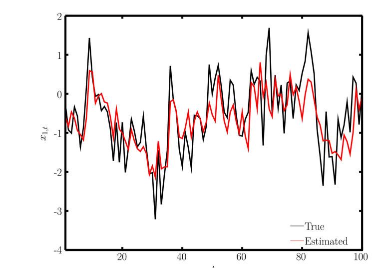
$$\left\{\overline{u}_{1:t+1}^{(m_u)}, \overline{x}_{1:t+1}^{(m_u)}\right\} \sim \chi_{t+1}, \text{ where } m_u = 1, \cdots, M_u.$$

Simulation results

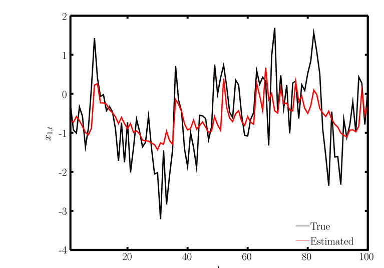
Estimation example



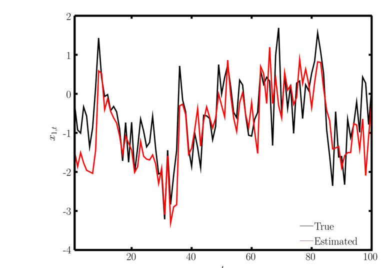
(a) All known parameters.



(b) σ_i^2 unknown.

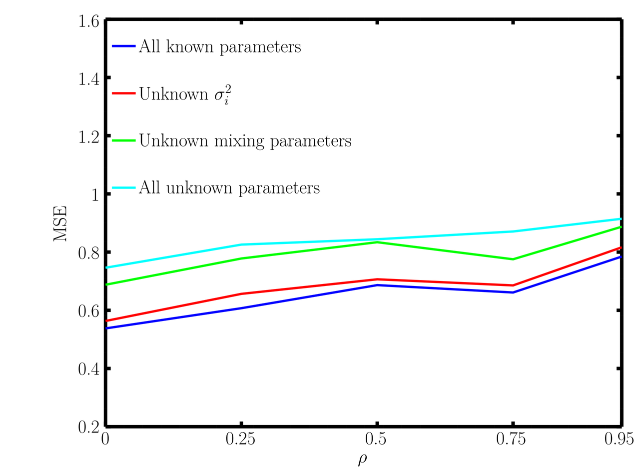


(c) Unknown A and C_ϵ.

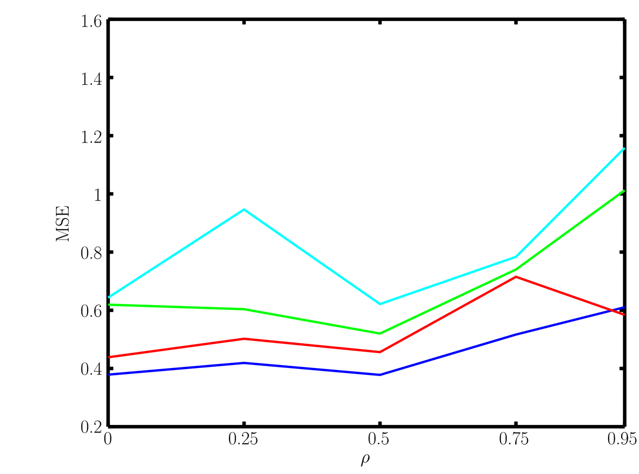


(d) All unknown.

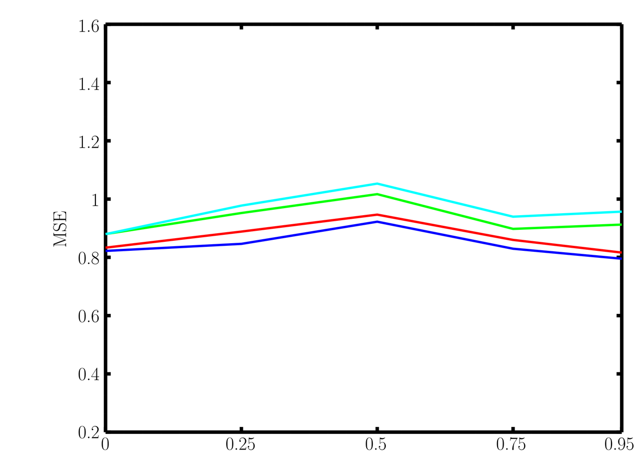
fGp filtering MSE



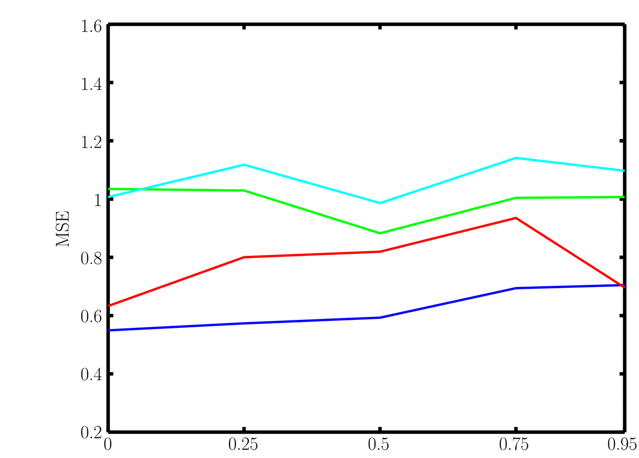
(e) u_1 for $H_1 = 0.7$, $H_2 = 0.7$.



(f) u_1 for $H_1 = 0.9$, $H_2 = 0.9$.

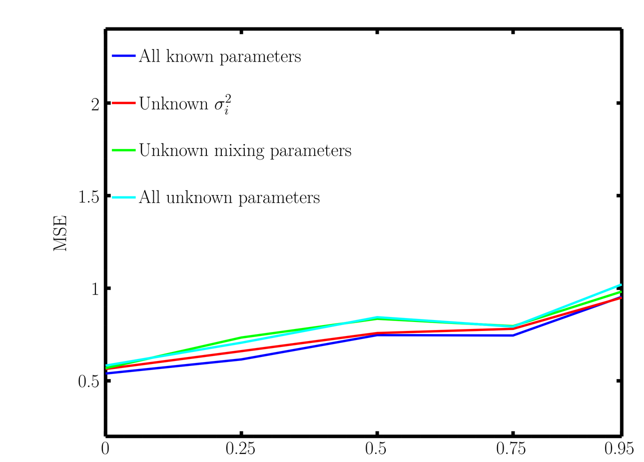


(g) u_2 for $H_1 = 0.7$, $H_2 = 0.7$.

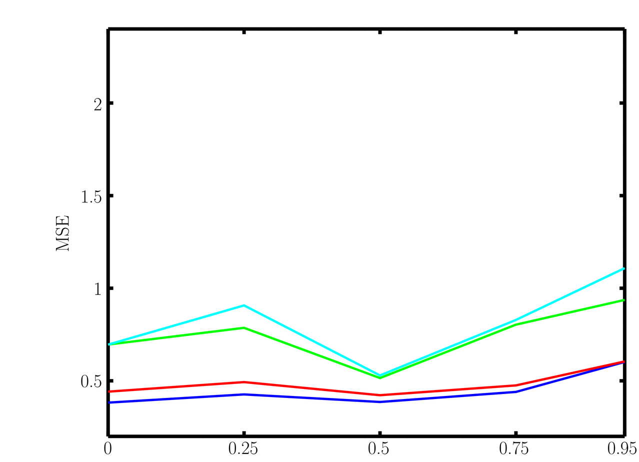


(h) u_2 for $H_1 = 0.9$, $H_2 = 0.9$.

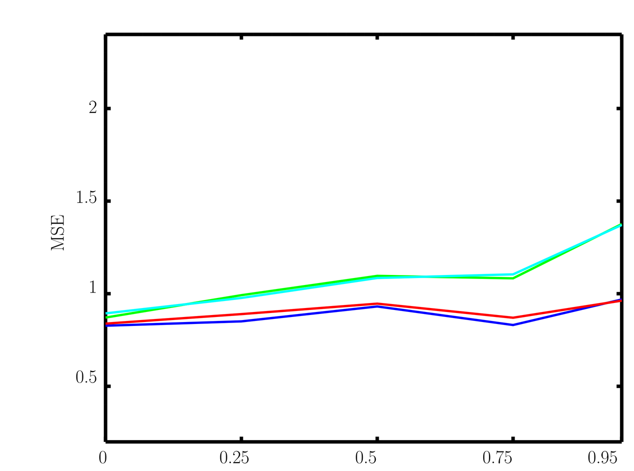
OfGp filtering MSE



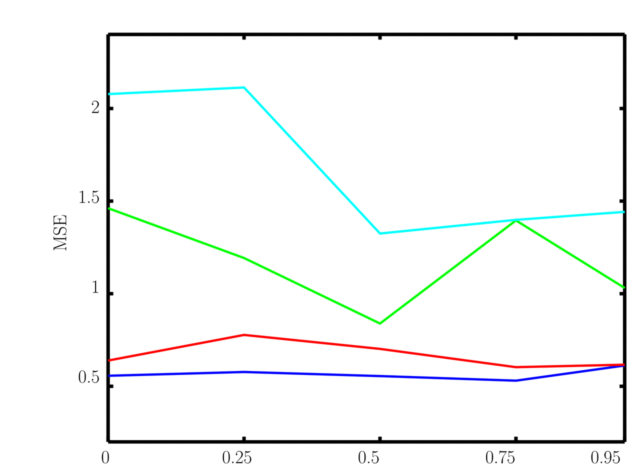
(i) x_1 for $H_1 = 0.7$, $H_2 = 0.7$.



(j) x_1 for $H_1 = 0.9$, $H_2 = 0.9$.



(k) x_2 for $H_1 = 0.7$, $H_2 = 0.7$.



(l) x_2 for $H_1 = 0.9$, $H_2 = 0.9$.