FAST SAMPLING OF GRAPH SIGNALS WITH NOISE VIA NEUMANN SERIES CONVERSION

Fen Wang[*], **Gene Cheung[+]**, Yongchao Wang[*]
[*]Xidian University, Xi'an, China ; [+]York University, Toronto, Canada

VISTA VISION: SCIENCE TO APPLICATIONS YORK UNIVERSITY

GRAPH SIGNAL PROCESSING

- Signals on irregular data kernels: $\mathcal{G} = (\mathcal{V}, \mathbf{W})$
- **Combinatorial Laplacian matrix**

$$\mathbf{L} = \mathbf{D} - \mathbf{W} \qquad d_i = \sum_{j=1}^{N} w_{i,j}$$

where degree matrix $\boldsymbol{D}$ is a diagonal matrix with entries

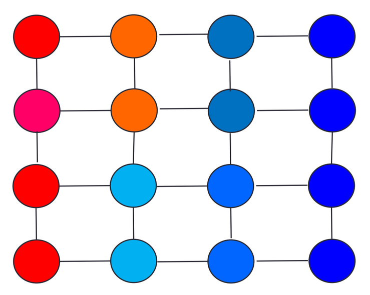


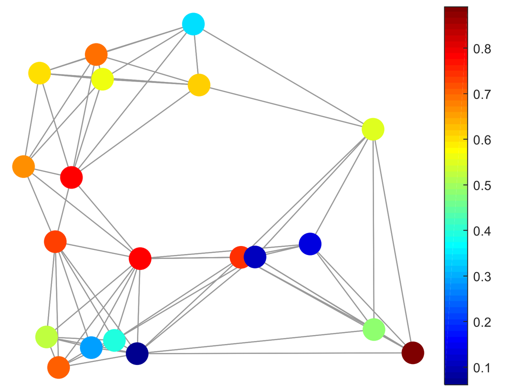
Image patch on 2D grid

- **Graph Fourier transform** (GFT)

$$\mathbf{L} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} \qquad \tilde{\mathbf{x}} = \mathbf{V}^{\mathrm{T}}\mathbf{x} \qquad \mathbf{x} = \mathbf{V}\tilde{\mathbf{x}}$$

Eigenvalue matrix (graph frequency); GFT matrix; Signal vector; GFT coefficients

GFT — Inverse GFT



Signal on a graph

- Bandlimited graph signal

The GFT coefficient $\tilde{x}$ are **non-zeros** only at the first K elements:

$$\mathbf{x} = \mathbf{V}_K \tilde{\mathbf{x}}_K$$

The first K columns of $\mathbf{V}$

SAMPLING OF NOISY BANDLIMITED GRAPH SIGNAL

- **Motivation:** sensing (acquiring samples) is expensive.
- **Goal:** sampling the ***most informative nodes*** for signal reconstruction.
- **Signal model:** ***noisy bandlimited*** graph signal
- Previous works:

Graph sampling
- Aggressive sampling
- Local measurement
- Selection sampling
  - Deterministic
    - Bandlimitedness
      - A-optimal (M. Tsitsvero.TSP2016; F. Wang. SPL2018)
      - E-optimal (S. Chen. TSP2015)
      - Spectral proxies (A. Anis. TSP2016; A. Anis. TIT2018)
      - Localized coverage (A. Sakiyama. TSP2019)
    - Smoothness: GMRF model (P. Chen. ICASSP2018; Y. Bai. ICASSP 2019)
  - Random selection

Previous works are not solving the A-optimality criterion directly without eigen-decomposition

Expected MSE value assuming i.i.d noise model with unit variance

Our goal

- Applications:

Sensor selection | *Active learning* | *Matrix completion*

AUGMENTED A-OPTIMAL GRAPH SAMPLING

- A-optimal sampling based on least square reconstruction
  - Noiseless observation: $\mathbf{x}_{\mathcal{S}} = \mathbf{C}\mathbf{V}_K\tilde{\mathbf{x}}_K$ — Perfect reconstruction: $\hat{\mathbf{x}} = \mathbf{V}_K(\mathbf{C}\mathbf{V}_K)^{\dagger}\mathbf{x}_{\mathcal{S}}$ (*Sampling operator corresponding to S*)
  - Noisy observation: $\mathbf{y}_{\mathcal{S}} = \mathbf{x}_{\mathcal{S}} + \mathbf{n}_{\mathcal{S}}$ — Minimal variance unbiased reconstruction: $\hat{\mathbf{x}} = \mathbf{V}_K(\mathbf{C}\mathbf{V}_K)^{\dagger}\mathbf{y}_{\mathcal{S}} = \mathbf{x} + \mathbf{V}_K(\mathbf{C}\mathbf{V}_K)^{\dagger}\mathbf{n}_{\mathcal{S}}$
  - Reconstruction MSE: $\|\hat{\mathbf{x}}-\mathbf{x}\|_2^2 = \mathrm{Tr}\left(\left[(\mathbf{C}\mathbf{V}_K)^{\mathrm{T}}\mathbf{C}\mathbf{V}_K\right]^{-1}\right)$
- Augmented A-optimal sampling objective

$$\mathbf{C}^* = \arg\min_{\mathbf{C}} \ \mathrm{Tr}\left(\left[(\mathbf{C}\mathbf{V}_K)^{\mathrm{T}}\mathbf{C}\mathbf{V}_K + \mu\mathbf{I}\right]^{-1}\right)$$

Small identity shift with $0 < \mu < 1$

- Neumann series theorem

If the absolute value of eigenvalues of A are all in the range (-1,1), then its Nuemann Series converges:

$$(\mathbf{I}-\mathbf{A})^{-1} = \sum_{l=0}^{\infty}\mathbf{A}^l$$

- Proposed objective function

$$\mathrm{Tr}\left(\left[(\mathbf{C}\mathbf{V}_K)^{\mathrm{T}}\mathbf{C}\mathbf{V}_K + \mu\mathbf{I}\right]^{-1}\right) = \frac{K-M}{\mu} + \mathrm{Tr}\left(\mathbf{T}_{\mathcal{S}} + \mu\mathbf{I}\right)^{-1}$$

Sample size $|\mathcal{S}| = M$

Submatrix of ideal graph filter $\mathbf{T} = \mathbf{V}_K\mathbf{V}_K^{\mathrm{T}}$

- Approximate objective function

$$\mathbf{C}^* = \arg\min_{\mathbf{C}} \ \mathrm{Tr}\left(\mathbf{T}_{\mathcal{S}}^{\mathrm{FGFT}} + \mu\mathbf{I}\right)^{-1}$$

- Chebyshev polynomial approximation
- Fast graph Fourier transform

RANK-1 UPDATE IN GREEDY ALGORITHM

- Minimize approximate objective via ***greedy*** sampling algorithm.
- Rank-1 update in each greedy step: reduce complexity from M^3 to M^2

$$\mathbf{G}_{\mathcal{S}\cup\{i\}}^{-1} = \begin{bmatrix} \mathbf{G}_{\mathcal{S}}^{-1} + a^{-1}\mathbf{G}_{\mathcal{S}}^{-1}\mathbf{g}_i\mathbf{g}_i^{\top}\mathbf{G}_{\mathcal{S}}^{-1} & -a^{-1}\mathbf{G}_{\mathcal{S}}^{-1}\mathbf{g}_i \\ -a^{-1}\mathbf{g}_i^{\top}\mathbf{G}_{\mathcal{S}}^{-1} & a^{-1} \end{bmatrix}$$

where $\mathbf{G} = \mathbf{T}^{\mathrm{FGFT}} + \mu\mathbf{I}$

TABLE I
COMPLEXITY COMPARISON OF DIFFERENT GRAPH SAMPLING STRATEGIES

	Preparation	Selection
Spectral Proxies	*NONE*	$\mathcal{O}(k\|\mathcal{E}\|MT_2(k) + NM)$
E-optimal	$\mathcal{O}((\|\mathcal{E}\|M + RM^3)T_1)$	$\mathcal{O}(NM^4)$
MFN	$\mathcal{O}((\|\mathcal{E}\|M + RM^3)T_1)$	$\mathcal{O}(NM^4)$
MIA	$\mathcal{O}(qN\|\mathcal{E}\|)$	$\mathcal{O}(NLM^{3.373})$
Proposed GFS	$\mathcal{O}(N^2\log^2 N)$	$\mathcal{O}(NM^3)$

- **No explicit full eigen-decomposition**
- **A-optimal related sampling objective**
- **Fast sampling in each greedy step**

e-mail: fenwang@stu.xidian.edu.cn;ychwang@mail.xidian.edu.cn;genec@yorku.ca

SHIFT PARAMETER DESIGN

- Design of μ based on inverse computation stability

Inverse of matrix $\mathbf{G}_s$ unstable if μ is extremely small since its eigenvalues are in $[\mu, 1+\mu]$. We propose to bound the condition number of $\mathbf{G}_s$

$$\kappa(\mathbf{G}_{\mathcal{S}}) = \frac{\lambda_{\max}(\mathbf{G}_{\mathcal{S}})}{\lambda_{\min}(\mathbf{G}_{\mathcal{S}})} \le \frac{1+\mu}{\mu} \le \kappa_0$$

Design μ based explicit condition number of G_S requires the information of S, thus we bound the worst case, which has no relation to S

- Reconstruction MSE of different μ

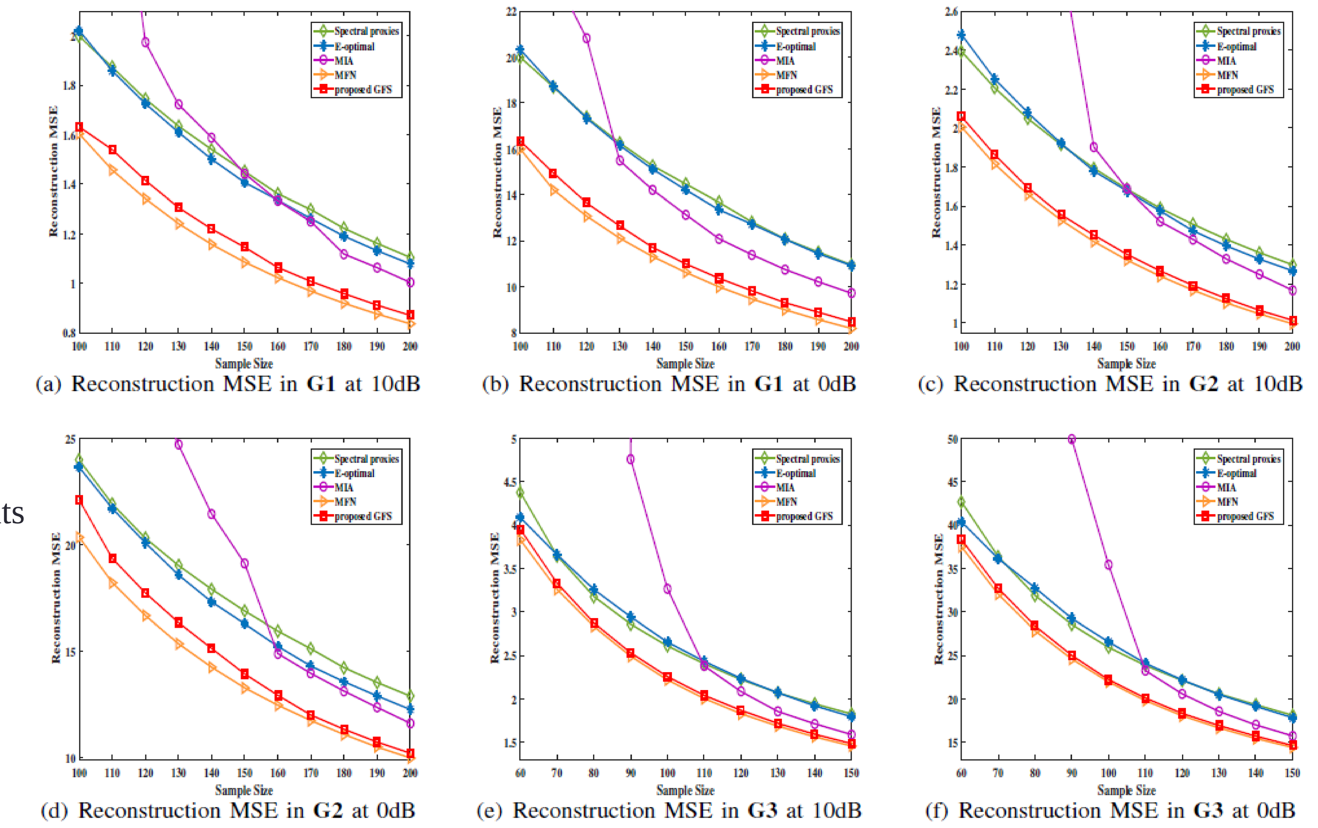
Table 1. Reconstruction MSE of Different μ at 0dB

Graph	μ	Sample size				
		100	110	120	130	140
G1	10^{-5}	16.10	14.55	13.43	12.44	11.63
	1/99	16.07	14.59	13.43	12.46	11.64
G2	10^{-5}	20.77	18.68	17.09	15.77	14.63
	1/99	20.77	18.73	17.12	15.78	14.64

- In experiments, we set $\kappa_0 = 100$. Reconstruction MSE is not sensitive to the choice of μ in community graph at 0dB.

EXPERIMENTAL RESULTS

- Experimental settings
  - **Graph model:**
    (G1) Community graphs with 1000 nodes and 31 communities;
    (G2) Sensor graphs with 1000 nodes;
    (G3) Hyper-cube graphs with 1002 nodes.
  - **Graph signal model:**
    (1) Bandwidth: $K = 50$
    (2) GFT coefficients: the non-zero coefficients are randomly generated from $N(1, 0.5^2)$; Coefficients after $K = 50$ are all zeros.
  - **Noise model:**
    Additional white Gaussian noise (AWGN) with different signal-to-noise ratios (SNRs).



(a) Reconstruction MSE in G1 at 10dB (b) Reconstruction MSE in G1 at 0dB (c) Reconstruction MSE in G2 at 10dB (d) Reconstruction MSE in G2 at 0dB (e) Reconstruction MSE in G3 at 10dB (f) Reconstruction MSE in G3 at 0dB

- Reconstruction MSE
  - Least square reconstruction
  - Different sampling methods

The proposed sampling method outperforms other three state-of-the-art sampling methods and approximates the MFN sampling in different sample size/ graphs/ SNRs with lower complexity .

REFERENCES

- G. Cheung, E. Magli, Y. Tanaka, and M. K. Ng, "Graph spectral image processing," *Proc. IEEE*, vol. 106, no.5, pp. 907–930, 2018.
- F. Wang, G. Cheung, and Y. C. Wang, "Low-complexity Graph Sampling with Noise and Signal Reconstruction via Neumann Series," Dec. 2018, arXiv:1812.02109.
- A. Anis, A. Gadde, and A. Ortega, "Efficient sampling set selection for bandlimited graph signals using graph spectral proxies," *IEEE Trans. Signal Process.*,, vol. 64, no. 14, pp. 3775–3789, 2016.
- S. Chen, R. Varma, A. Sandryhaila, and J. Kovačević, "Discrete signal processing on graphs: Sampling theory," *IEEE Trans. Signal Process.*, vol. 63, no. 24, pp. 6510-6523, 2015.
- Shomorony and A. Avestimehr, "Sampling large data on graphs," in *Proc. IEEE Global Conf. Signal Inf. Process. (GlobalSIP)*, pp. 933-936, Dec. 2014.
- F. Wang, Y. C. Wang and G. Cheung, "A-optimal sampling and robust reconstruction for graph signals via truncated Neumann series," *IEEE Signal Processing Letters*, vol. 25, no. 5, pp. 680-684, May. 2018.