3D Coprime Arrays in Sparse Sensing

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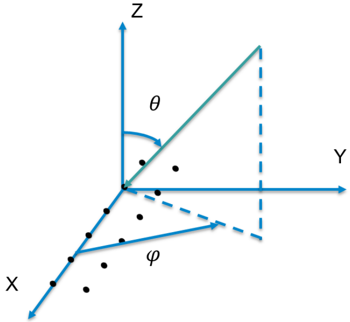
DOA Estimation[1]



Interferences between dense antennas.

Uncorrelated sources

↓

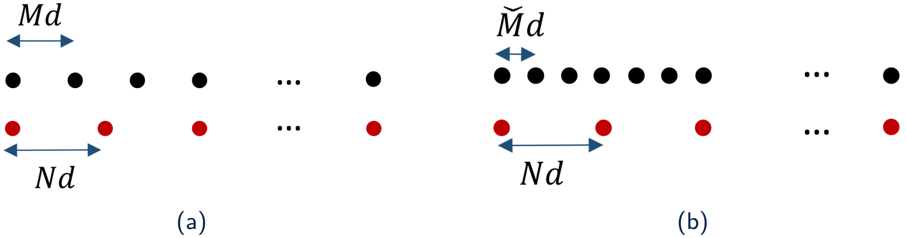
Sensor Array

↓

Estimation of θ and φ

[1] Van Trees HL. Optimum array processing: Part IV of detection, estimation, and modulation theory. John Wiley Sons; 2004.

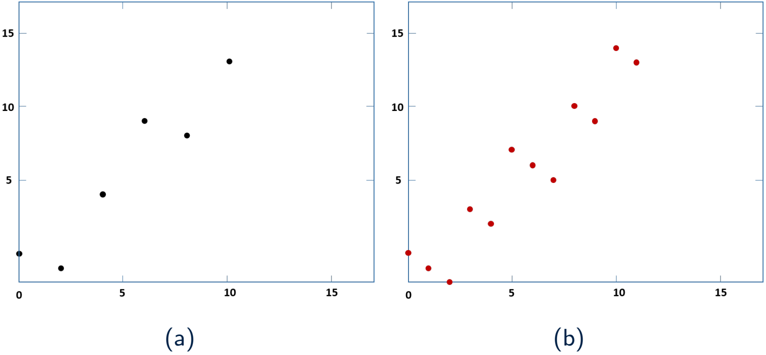
Examples of 1D Coprime Arrays



(a) The coprime array[2]. (b) The CACIS configuration[3].

[2]P.P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," IEEE Trans. Signal Process., vol. 59, no. 2, pp. 573–586, Feb. 2011.
[3]S. Qin, Y. D. Zhang, and M.G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," IEEE Trans. Signal Process., vol. 63, pp. 1377–1390, Mar. 2015.

Examples of 2D Coprime Arrays[2]



(a) $\mathbf{M}\mathbf{n}_1$ and (b) $\mathbf{N}\mathbf{n}_2$ where $\mathbf{M} = [2, 2; -1, 5], \mathbf{N} = [1, 2; -1, 4], \mathbf{n}_1 = \text{FPD}(\mathbf{N})$, and $\mathbf{n}_2 = \text{FPD}(\mathbf{M})$.

[4] P.P. Vaidyanathan and P. Pal, Theory of sparse coprime sensing in multiple dimensions. IEEE Trans. Signal Process., vol. 59, no. 8, pp. 3592–3608, Aug. 2011.

Coprime Matrices

Definition 1 (Coprime Matrices)

Two integer matrices $\mathbf{B}_m$ and $\mathbf{B}_n$ are left coprime if and only if there exist integer matrices $\mathbf{C}$ and $\mathbf{D}$ such that

$$\mathbf{B}_m\mathbf{C} + \mathbf{B}_n\mathbf{D} = \mathbf{I}. \tag{1}$$

Our Goal

Find a more general and systematic way of finding pairwise 3-by-3 coprime matrices.

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Cubic Integers

Definition 2 (Cubic Field)

A field extension of degree 3 over rational numbers $\mathbb{Q}$, i.e., it is a $\mathbb{Q}$-vector space of dimension three.

E.g. $\gamma = 5 + 2\sqrt[3]{12} + \sqrt[3]{12^2}$ is in the ring of integers of the cubic field $\mathbb{Q}(\sqrt[3]{12})$.

$a + b\sqrt[3]{12} + c\sqrt[3]{12^2}$ is an cubic integer in $\mathbb{Q}(\sqrt[3]{12})$ $(a, b, c \in \mathbb{Z})$.

Other algebraic integers include quadratic integers, which can be used for planar arrays[5].

[5] C. Li, L. Gan and C. Ling, "Coprime sensing via Chinese remaindering over quadratic fields—Part I: Array designs," in IEEE Trans. Signal Process., vol. 67, no. 11, pp. 2898-2910, June 2019.

Integral Basis of Cubic Field

A pure cubic field K is a field extension of $\mathbb{Q}$ in the form of $\mathbb{Q}(\sqrt[3]{r})$ where r is a non-unit cubic-free integer. An integral basis of $\mathbb{Q}(\theta)$ $(\theta = \sqrt[3]{r},\ r = ab^2)$ is

- if $r \not\equiv \pm 1 \pmod 9$

$$\left\{1,\ \theta,\ \frac{\theta^2}{b}\right\}, \quad \text{or} \tag{2}$$

- if $r \equiv \pm 1 \pmod 9$

$$\left\{1,\ \theta,\ \frac{\theta^2 + r\theta + b^2}{3b}\right\}. \tag{3}$$

Representation Matrix of a Cubic Integer

By stacking the coefficients corresponding to basis (2), the matrix representation of $m = m_1 + m_2\theta + m_3\frac{\theta^2}{b}$ is

$$\mathbf{B}_m = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_3 ab & m_1 & m_2 b \\ m_2 ab & m_3 a & m_1 \end{pmatrix}, \tag{4}$$

E.g. The corresponding matrix of $\gamma = 5 + 2\sqrt[3]{12} + \sqrt[3]{12^2}$ is

$$\mathbf{B}_\gamma = \begin{pmatrix} 5 & 2 & 2 \\ 12 & 5 & 4 \\ 12 & 6 & 5 \end{pmatrix}. \tag{5}$$

Review of Chinese Remainder Theorem

If $\mathcal{I}$ and $\mathcal{J}$ are coprime, the Chinese Remaindering Theorem asserts that:

$$R/\mathcal{I}\mathcal{J} \simeq R/\mathcal{I} \times R/\mathcal{J}. \tag{6}$$

For all $a_k \in R/\mathcal{I}$ and $b_j \in R/\mathcal{J}$, it can be verified that every pair (a_k, b_j) forms the solution

$$z \equiv x_k b_j + y_j a_k \pmod{\mathcal{I}\mathcal{J}}. \tag{7}$$

where $x_k \in \mathcal{I}$ and $y_j \in \mathcal{J}$.
Eg. $R = \mathbb{Z}$ and $\mathcal{I} = \langle 3 \rangle = 3\mathbb{Z} = \pm 3, \pm 6 \cdots$. Let $\mathcal{J} = \langle 5 \rangle$, thus $z \equiv x_k b_j + y_j a_k \pmod{15}$, where $x_k \in \langle 3 \rangle$, $b_j = \mathbb{Z}/5\mathbb{Z}$, $y_j \in \langle 5 \rangle$, and $a_k = \mathbb{Z}/3\mathbb{Z}$.

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Algebraic Construction of 3D Lattices

Definition 3 (Lattice)

Given n linearly independent column vectors $\mathbf{g}_1, \cdots \mathbf{g}_n$ of dimension n, an nD lattice Λ is defined as

$$\Lambda = \left\{ \sum_{k=1}^{n} x_k \mathbf{g}_k : x_k \in \mathbb{Z} \right\}. \quad (8)$$

The set $\{\mathbf{g}_1, \cdots \mathbf{g}_n\}$ is called the basis and the matrix that consists of this basis is the generator matrix of Λ which can be written as

$$\mathbf{G} = [\mathbf{g}_1 | \cdots | \mathbf{g}_n]. \quad (9)$$

CRT Arrays

Definition 4 (CRT array)

$\mathbf{B}_m$ and $\mathbf{B}_n$ are the matrices that represent two coprime cubic integers m and n respectively. A CRT array comprises two subarrays:

$$\mathcal{S}_1 = \{\mathbf{z}_m : \mathbf{z}_m = \mathbf{B}_m \mathbf{x}_2\}, \text{ and } \mathcal{S}_2 = \{\mathbf{z}_n : \mathbf{z}_n = \mathbf{B}_n \mathbf{x}_1\}.$$

The difference coarray:

$$\mathcal{S} = \{\mathbf{z}_m - \mathbf{z}_n : \mathbf{z}_m \in \mathcal{S}_1, \mathbf{z}_n \in \mathcal{S}_2\}. \tag{10}$$

Generalized Chinese Remainder Theorem asserts the surged gain of degrees of freedom.

Algebraic Conjugates

$m \in \mathbb{Q}(\theta)$ can also be expressed as

$$m = u_1 + u_2\theta + u_3\theta^2. \tag{11}$$

Let $\omega = e^{j2\pi/3}$. The three embeddings that map $m \in \mathcal{O}_K$ into $\mathbb{C}$ are:

$$\begin{aligned} m &\to m = u_1 + u_2\theta + u_3\theta^2, \\ m &\to m' = u_1 + u_2\omega\theta + u_3\omega^2\theta^2, \\ m &\to m'' = u_1 + u_2\omega^2\theta + u_3\omega\theta^2. \end{aligned} \tag{12}$$

The norm of m is defined as

$$\mathrm{N}(m) = \det(\mathbf{B}_m) = mm'm''. \tag{13}$$

Coprimality of Cubic Integers and Their Matrices

Lemma 1: Two cubic integers are coprime if and only if their corresponding matrices are coprime.

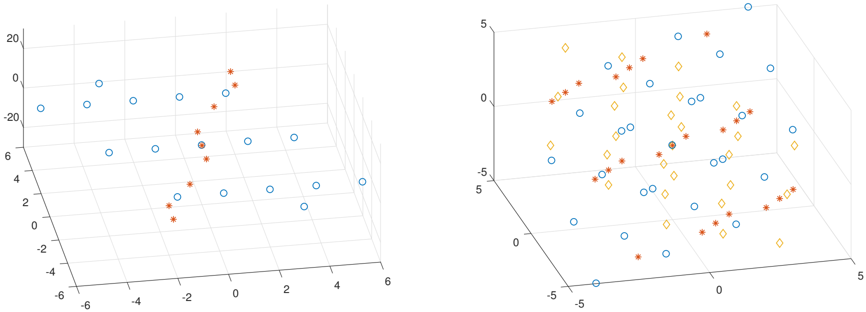
Theorem 1

In a pure cubic field, m and $\hat{m}$ are coprime if and only if

$$GCD(N(m)\,,\,3\hat{u}_1) = 1, \tag{14}$$

where $\hat{m} = m'm''; 3\hat{u}_1 = 3m_2m_3ab - 3m_1^2$, *if* $r \not\equiv \pm 1 \pmod 9$, *and* $3\hat{u}_1 = 3m_1^2 + (2m_1 - m_2a)m_3b + m_3^2b^2(1-a^2)/3$ *otherwise* .

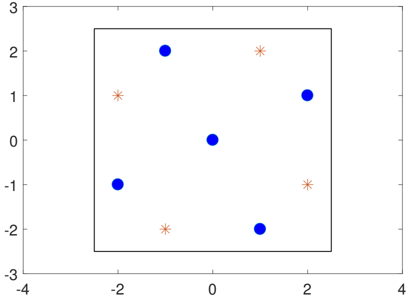
Examples of 3D CRT Arrays



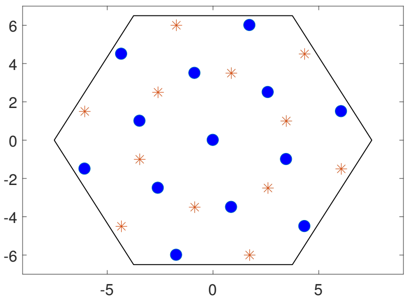
(a) (b)

(a) $\langle\, 5 + 2\sqrt[3]{12} + \sqrt[3]{12^2}\,\rangle$ (red stars) and $\langle\, 1 + 2\sqrt[3]{12} - \sqrt[3]{12^2}\,\rangle$ (blue dots) in $\mathbb{Z}[\sqrt[3]{12}]$.
(b) $\langle\, 3 + \sqrt[3]{4}\,\rangle$ (red stars), $\langle\, -1 + 2\sqrt[3]{2} + \sqrt[3]{4}\,\rangle$ (blue dots), and $\langle\, 1 - 2\sqrt[3]{4}\,\rangle$ (yellow diamonds) in $\mathbb{Z}[\sqrt[3]{2}]$.

Examples of 2D CRT Arrays



(a)



(b)

(a) $\langle 2+i \rangle$ (red stars) and $\langle 2-i \rangle$ (blue dots) in $\mathbb{Z}[i]$.
(b) $\langle 3+2i \rangle$ (red stars) and $\langle 3-2i \rangle$ (blue dots) in $\mathbb{Z}[\omega]$.

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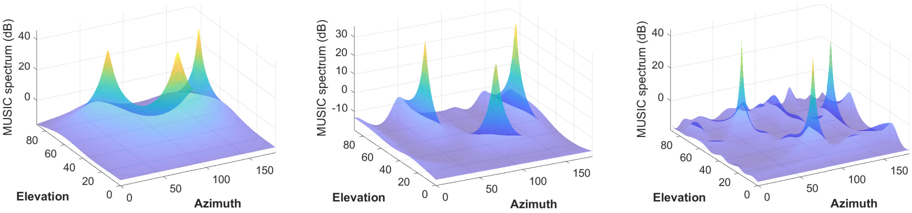
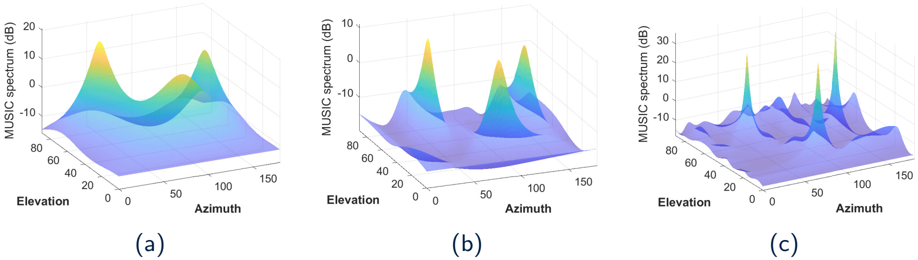
DOA Estimation without Mutual Coupling Effect



(a) (b) (c)

MUSIC spectra without considering mutual coupling effect. (a) 3D uniformly distributed array; (b) 2D CRT array: the quadratic field is $\mathbb{Q}(i)$; $p = 2 + i$; $q = 2 - i$. (c) 3D CRT array: the cubic field is $\mathbb{Q}(\sqrt[3]{2})$; $m = 3 + \sqrt[3]{4}$; $n = -1 + 2\sqrt[3]{2} + \sqrt[3]{4}$.

DOA Estimation with Mutual Coupling Effect



MUSIC spectra with the presence of mutual coupling effect. (a) 3D uniformly distributed array; (b) 2D CRT array; (c) 3D CRT array.

[6] B. Friedlander and AJ. Weiss, "Direction finding in the presence of mutual coupling," in IEEE Trans. on antennas and propagation, vol. 39, no. 3, pp. 273-84, Mar. 1991.

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Concluding Remarks

- A new class of 3D coprime array is presented. They enjoy the sparse geometry and a surged gain of DOF.
- In the future, more compact 3D arrays will be developed based on denser lattices, and more applications of CRT arrays will be presented with comparisons to conventional 3D methods.

Thank you!