Adaptive Subspace Detector in High Dimensional Space with Insufficient Training Data

Aref Miri Rekavandi, Abd-Krim Seghouane, Robin J. Evans

Department of Electrical and Electronic Engineering

Melbourne School of Engineering, The University of Melbourne, Australia

THE UNIVERSITY OF MELBOURNE

Introduction

- When parameters of the noise distribution are known, the likelihood ratio test is the uniformly most powerful (UMP) test [2].
- Adaptive subspace detectors (ASD) generalize matched subspace detectors (MSD) by accounting for possible correlation [1].
- For a limited training data, applying ASD directly on data is inefficient.
- Here we propose a detector which works in lower dimensional space and improves the performance.

Problem Definition

Given d samples from a real and scalar time series $\mathbf{y}(i), i = 0, 1, ..., d-1$ that is represented by the column vector $\mathbf{y}$. This vector of observations is generated by some components based on a general linear model (GLM):

$$\mathbf{y} = \mathbf{H}\boldsymbol{\theta} + \mathbf{n} \tag{1}$$

where

- $\mathbf{H} \in \mathbb{R}^{d \times p}$ is a known full rank matrix.
- $\boldsymbol{\theta} \in \mathbb{R}^{p}$ is unknown coordinates.
- $\mathbf{n} \sim \mathcal{N}(0, \sigma^2 \mathbf{R})$, where σ^2 and $\mathbf{R}$ are unknown.
- A traning set of K signal free observations is available.

The aim of detection is to decide between the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} = \mathbf{0}$ and the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\theta} \neq \mathbf{0}$ for a given observation $\mathbf{y}$.

The Proposed Algorithm

Goal of the algorithm is to bring observed data to a lower dimensional space to decrease the number of unknown parameters and increase the estimation accuracy. To achieve this goal, we assume:

$$\mathbf{y} = \mathbf{U}_{d_1}\mathbf{z} \quad \text{and} \quad \mathbf{n} = \mathbf{U}_{d_1}\mathbf{e} \tag{2}$$

where $\mathbf{z} \in \mathbb{R}^{d_1}$ and $\mathbf{e} \in \mathbb{R}^{d_1}$ with $d_1 \ll d$ and $\mathbf{U}_{d_1}$ is an unknown $d \times d_1$ matrix whose columns are orthonormal, (i.e. $\mathbf{U}_{d_1}^T \mathbf{U}_{d_1} = \mathbf{I}_{d_1}$. Now the GLM is

$$\mathbf{z} = \mathbf{U}_{d_1}^T \mathbf{H}\boldsymbol{\theta} + \mathbf{e} \tag{3}$$

where $\mathbf{z}$ is in lower dimensional space and subspace of the signal is $\mathbf{U}_{d_1}^T \mathbf{H}$, instead of $\mathbf{H}$. To solve the problem, we need to find the best set of projection basis and then apply the well known ASD on the latent variable $\mathbf{z}$.

1. Estimation of $\mathbf{U}_{d_1}$: For finding the best projection matrix, we consider two different cases of covariance structures:
   - $\mathbf{R}$ with uniform diagonal components: In this case we need to solve the following optimization problem:

     $$\hat{\mathbf{U}}_{d_1} = \arg\min_{\mathbf{U}_{d_1}} \|\mathbf{U}_{d_1}\mathbf{U}_{d_1}^T \mathbf{S} \mathbf{U}_{d_1}\mathbf{U}_{d_1}^T - \mathbf{S}\|_F^2 \tag{4}$$

     where $\|.\|_F^2$ shows Frobenius norm of a matrix. This optimization problem means that we need to regularize the sample covariance in a way that the result gets close enough to the sample covariance $\mathbf{S}$. For a known d_1 the solution is the first d_1 eigen-vectors of $\mathbf{S}$.
   - Ill-conditioned $\mathbf{R}$: In this case study, we need to solve the following optimization problem:

     $$\hat{\mathbf{U}}_{d_1} = \arg\min_{\mathbf{U}_{d_1}} |\mathbf{U}_{d_1}^T \mathbf{S} \mathbf{U}_{d_1}| + \alpha \|\mathbf{U}_{d_1}\mathbf{U}_{d_1}^T \mathbf{S} \mathbf{U}_{d_1}\mathbf{U}_{d_1}^T - \mathbf{S}\|_F^2 \tag{5}$$

     which tries to maximize the likelihood such that the estimated covariance remains in the neighborhood of the sample covariance. The combination of the first a and the last b eigen-vectors of $\mathbf{S}$ is the solution of problem (5) where $a + b = d_1$.
2. Detection: The ASD for the latent variable $\mathbf{z}$ is:

   $$l(\mathbf{z}) = \frac{\mathbf{z}^T \hat{\boldsymbol{\Lambda}}^{-\frac{1}{2}} \mathbf{P}_{\mathbf{B}} \hat{\boldsymbol{\Lambda}}^{-\frac{1}{2}} \mathbf{z}}{\mathbf{z}^T \hat{\boldsymbol{\Lambda}}^{-1} \mathbf{z}} \tag{6}$$

   where $\mathbf{B}$ is $\hat{\boldsymbol{\Lambda}}^{-\frac{1}{2}} \mathbf{U}_{d_1}^T \mathbf{H}$ and $\hat{\boldsymbol{\Lambda}} = \mathbf{U}_{d_1}^T \mathbf{S} \mathbf{U}_{d_1}$. We name our reduced version of ASD, the adaptive reduced subspace detector (ARSD).

Algorithm Overview

Input: $\mathbf{y}$, $\mathbf{N}$, $\mathbf{H}$, d_1, α

Estimate $\mathbf{S}$ and find its eigen-vectors by **SVD**.

Find the optimal basis functions by solving equations (4) or (5).

Find the latent variable $\mathbf{z}$ and new whitened signal subspace $\mathbf{B}$.

Apply well known ASD on the $\mathbf{z}$.

Output: $\mathcal{H}_0$ or $\mathcal{H}_1$

Simulation Results

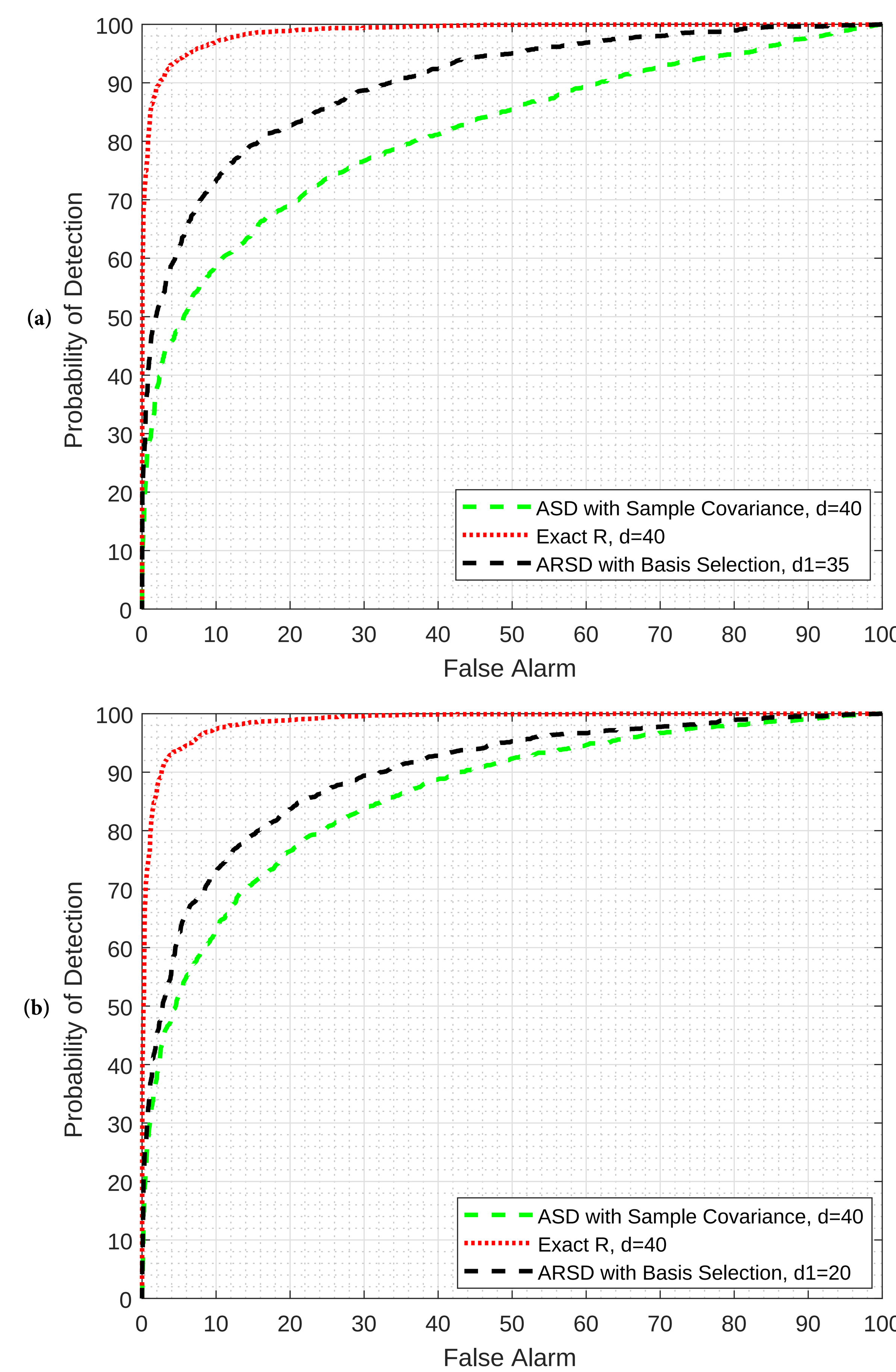

Figure 1: ROC of detectors in SNR=12 dB (a) $\mathbf{R}$ with uniform diagonal components, (b) Ill-conditioned $\mathbf{R}$

Simulation Results Cntd.

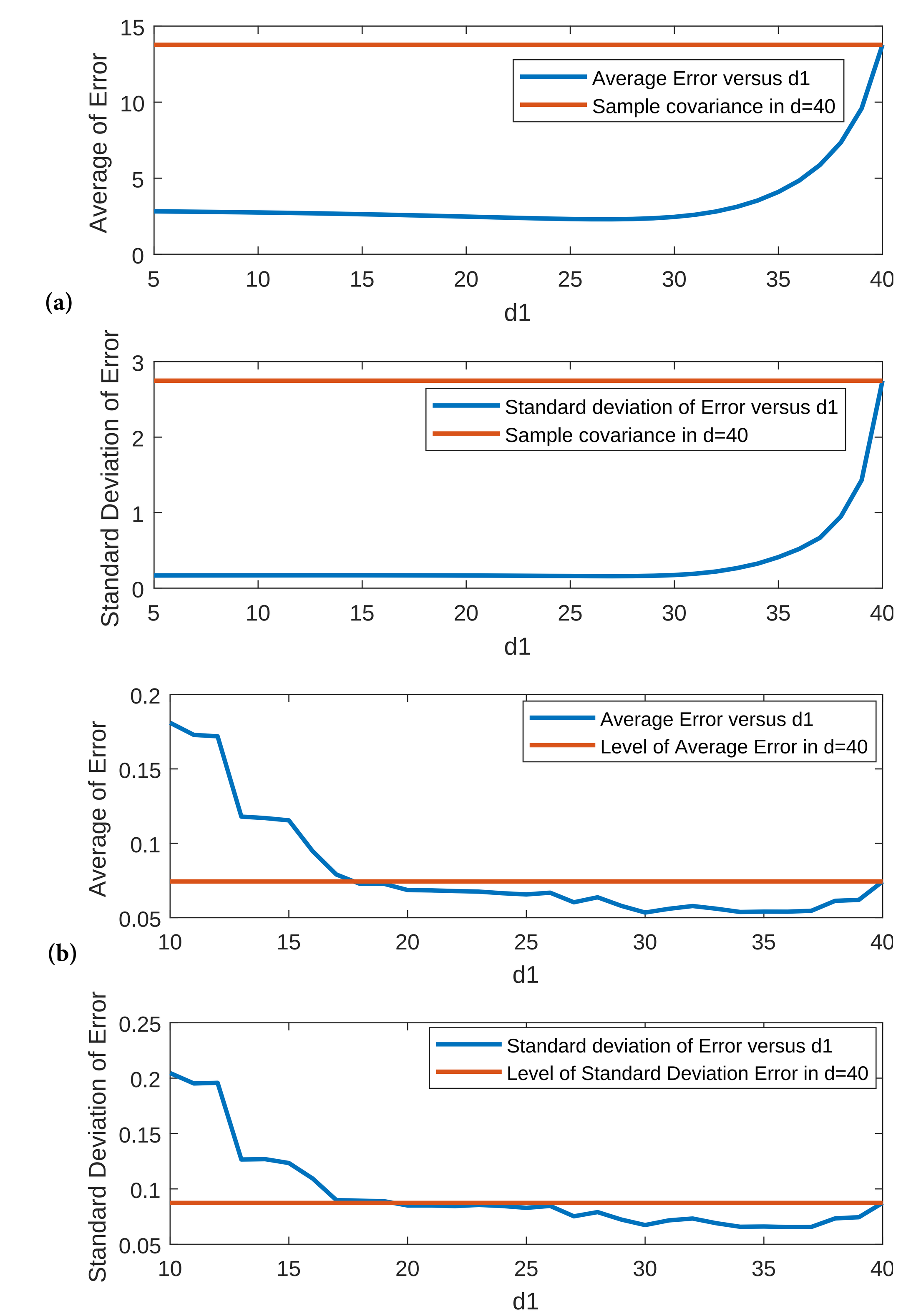

Figure 2: Average and standard deviation of Frobenius norm of difference between the actual and the estimated (a) covariance inverse and (b) $\boldsymbol{\theta}$.

References

[1] Shawn Kraut, Louis L Scharf, and L Todd McWhorter. Adaptive subspace detectors. *IEEE Transactions on signal processing,* 49(1):1–16, 2001.

[2] Jerzy Neyman and Egon S Pearson. IX. on the problem of the most efficient tests of statistical hypotheses. *Phil. Trans. R. Soc. Lond. A,* 231(694-706):289–337, 1933.