Pairwise Approximate K-SVD

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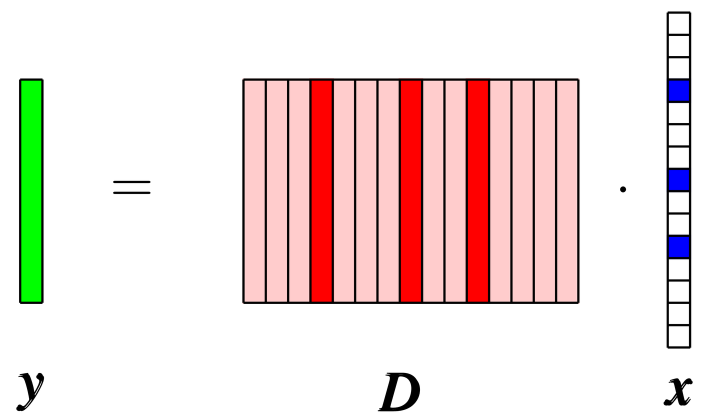
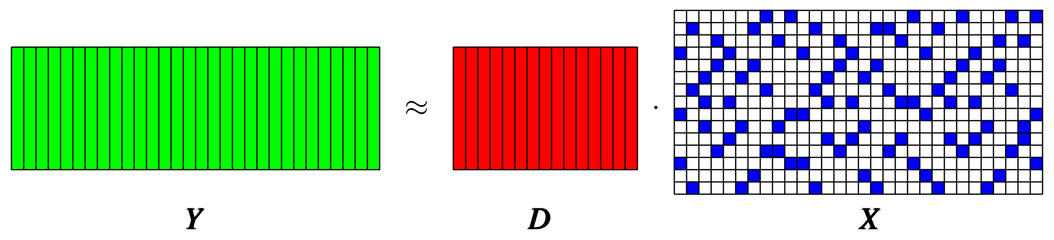
Dictionary Learning

Sparse representation of vectorized signal $\boldsymbol{y} \in \mathbb{R}^{m^2}$

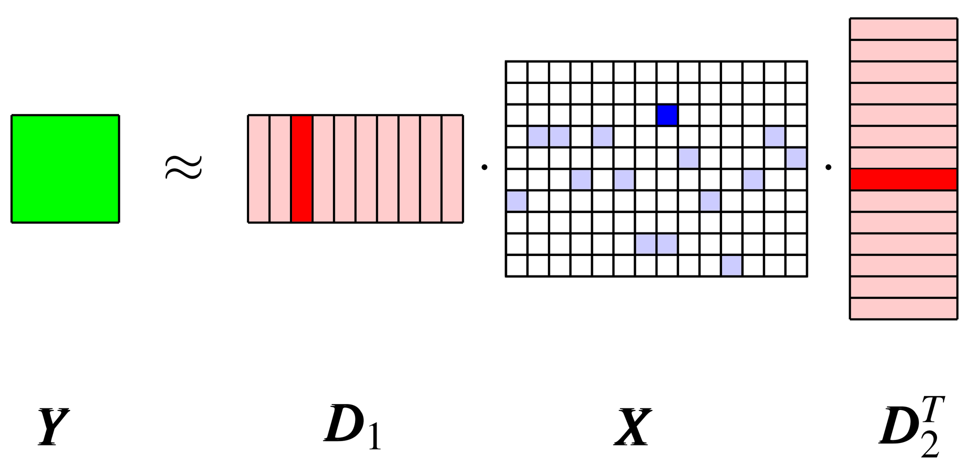


- $\boldsymbol{D} \in \mathbb{R}^{m \times n_1 n_2}$ is the dictionary
- $\boldsymbol{x} \in \mathbb{R}^{n_1 n_2}$ is the s-sparse representation
- dictionary columns are called atoms

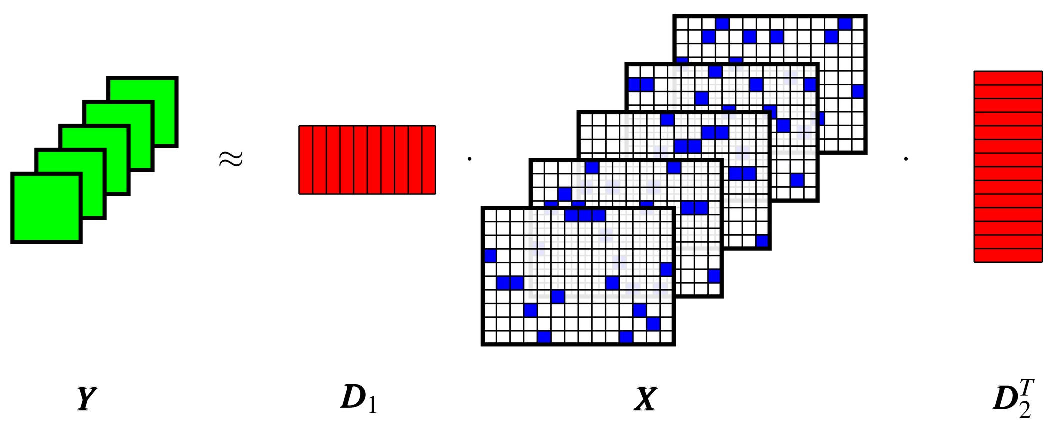


Separable Dictionary Learning

Sparse representation of a 2D signal $\boldsymbol{Y} \in \mathbb{R}^{m \times m}$



- $\boldsymbol{D}_1 \in \mathbb{R}^{m \times n_1}$, $\boldsymbol{D}_2 \in \mathbb{R}^{m \times n_2}$ left and right dictionaries
- $\boldsymbol{X} \in \mathbb{R}^{n_1 \times n_2}$ is the s-sparse representation



First algorithm SeDiL: highly complex algorithm based on manifold gradient search. Alternatives:

- rank-1 CP decomposition based (2DS), closer to K-SVD
- tensor orientated, using t-SVD (K-TSVD, TKSVD)
- non-separable focused on the Kronecker operation (SuKro, 2DNS)

Optimization Problem

Goal: Minimize the error

$$\min_{\boldsymbol{D}_1,\boldsymbol{D}_2,\boldsymbol{X}} \sum_{k=1}^{N} \|\boldsymbol{Y}_k - \boldsymbol{D}_1 \boldsymbol{X}_k \boldsymbol{D}_2^T\|_F^2 \quad \text{subject to } \|\boldsymbol{X}_k\|_0 \le s,\ k = 1:N;\quad \|\boldsymbol{d}_{1i}\|_2 = 1,\ i = 1:n_1;\quad \|\boldsymbol{d}_{2j}\|_2 = 1,\ j = 1:n_2 \tag{1}$$

Problem (1) is DL with dictionary $\boldsymbol{D}_2 \otimes \boldsymbol{D}_1$:

$$\|\boldsymbol{Y}_k - \boldsymbol{D}_1 \boldsymbol{X}_k \boldsymbol{D}_2^T\|_F^2 = \|\mathrm{vec}\,\boldsymbol{Y}_k - (\boldsymbol{D}_2 \otimes \boldsymbol{D}_1)\mathrm{vec}\,\boldsymbol{X}_k\|^2 \tag{2}$$

For the left and right atom indices $\mathcal{I} = \{i_1, \dots, i_s\}$ and $\mathcal{J} = \{j_1, \dots, j_s\}$, the approximation of $\boldsymbol{Y}_k$ is

$$\|\mathcal{E}(\boldsymbol{X}_k)\|_F = \|\boldsymbol{Y}_k - \sum_{\iota=1}^{s} x^{(k)}_{i_\iota j_\iota} \boldsymbol{d}_{1i_\iota} \boldsymbol{d}_{2j_\iota}^T\|_F = \|\mathrm{vec}\,\boldsymbol{Y}_k - \sum_{\iota=1}^{s} (\boldsymbol{d}_{2j_\iota} \otimes \boldsymbol{d}_{1i_\iota}) x^{(k)}_{i_\iota, j_\iota}\|. \tag{3}$$

TKSVD Dictionary Update

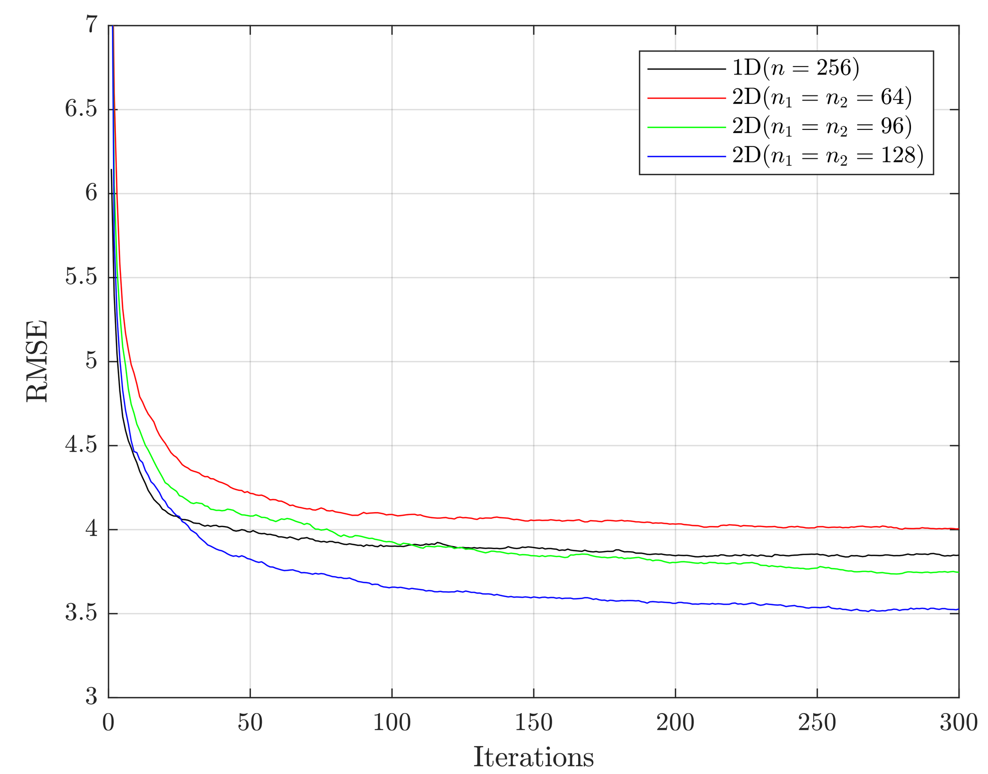
Simultaneously updates each (i, j) atoms pair

$$\boldsymbol{R}_k = \mathcal{E}(\boldsymbol{X}_k) + x^{(k)}_{i,j} \boldsymbol{d}_{1i} \boldsymbol{d}_{2j}^T \tag{4}$$

$$\min_{\boldsymbol{d}_{1i}, \boldsymbol{d}_{2j}} \sum_{k=1}^{N} \left\| \boldsymbol{R}_k - x^{(k)}_{i,j} \boldsymbol{d}_{1i} \boldsymbol{d}_{2j}^T \right\|_F^2 \quad \text{subject to } \|\boldsymbol{d}_{1i}\|_2 = 1, \|\boldsymbol{d}_{2j}\|_2 = 1. \tag{5}$$

- many $x^{(k)}_{i,j}$ are zero; rank-1 CP decomposition
- process iterated by TKSVD for all atom pairs
- diverges when two $\boldsymbol{d}_{2j}$ and $\boldsymbol{d}_{2j'}$ pair with $\boldsymbol{d}_{1i}$

RMSE



Pairwise Approximate K-SVD

Fix everything in (1) except for either atom i of $\boldsymbol{D}_1$

$$\min_{\boldsymbol{d}_{1i}} \sum_{k=1}^{N} \left\| \boldsymbol{R}_{1k} - \boldsymbol{d}_{1i} \boldsymbol{x}^{(k)}_{i,\mathcal{P}_k} \boldsymbol{D}_{2\mathcal{P}_k}^T \right\|_F^2 \text{ s.t. } \|\boldsymbol{d}_{1i}\|_2 = 1 \tag{6}$$

or atom j of $\boldsymbol{D}_2$

$$\min_{\boldsymbol{d}_{2j}} \sum_{k=1}^{N} \left\| \boldsymbol{R}_{2k} - \boldsymbol{D}_{1\mathcal{Q}_k} \boldsymbol{x}^{(k)}_{\mathcal{Q}_k,j} \boldsymbol{d}_{2j}^T \right\|_F^2 \text{ s.t. } \|\boldsymbol{d}_{2j}\|_2 = 1$$

When representing $\boldsymbol{Y}_k$

- $\mathcal{P}_k$ are the atoms in $\boldsymbol{D}_2$ pairing with atom i in $\boldsymbol{D}_1$
- $\mathcal{Q}_k$ are the atoms in $\boldsymbol{D}_1$ pairing with $\boldsymbol{d}_{2j}$

The k-th term of the sum from (6) becomes

$$\|\boldsymbol{R}_{1k}\|_F^2 - 2\mathrm{tr}(\boldsymbol{R}_{1k}^T \boldsymbol{d}_{1i} \boldsymbol{x}^{(k)}_{i,\mathcal{P}_k} \boldsymbol{D}_{2\mathcal{P}_k}^T) + \boldsymbol{x}^{(k)}_{i,\mathcal{P}_k} \boldsymbol{D}_{2\mathcal{P}_k}^T \boldsymbol{D}_{2\mathcal{P}_k} (\boldsymbol{x}^{(k)}_{i,\mathcal{P}_k})^T, \tag{7}$$

Minimized when the trace is maximized

$$\max_{\boldsymbol{d}_{1i}} \left(\sum_{k=1}^{N} \boldsymbol{x}^{(k)}_{i,\mathcal{P}_k} \boldsymbol{D}_{2\mathcal{P}_k}^T \boldsymbol{R}_{1k}^T \right) \boldsymbol{d}_{1i} \text{ s.t. } \|\boldsymbol{d}_{1i}\|_2 = 1 \tag{8}$$

Maximum when $\boldsymbol{d}_{1i}$ is collinear to the sum vector

$$\boldsymbol{d}_{1i} = \frac{\boldsymbol{t}_1}{\|\boldsymbol{t}_1\|},\ \boldsymbol{t}_1 = \sum_{k=1}^{N} \boldsymbol{R}_{1k} \boldsymbol{D}_{2\mathcal{P}_k} (\boldsymbol{x}^{(k)}_{i,\mathcal{P}_k})^T. \tag{9}$$

$$(\boldsymbol{x}^{(k)}_{i,\mathcal{P}_k})^T = (\boldsymbol{D}_{2\mathcal{P}_k}^T \boldsymbol{D}_{2\mathcal{P}_k})^{\dagger} \boldsymbol{D}_{2\mathcal{P}_k}^T \boldsymbol{R}_{1k}^T \boldsymbol{d}_{1i}. \tag{10}$$

For atom $\boldsymbol{d}_{2j}$ we update following

$$\boldsymbol{d}_{2j} = \frac{\boldsymbol{t}_2}{\|\boldsymbol{t}_2\|},\ \boldsymbol{t}_2 = \sum_{k=1}^{N} \boldsymbol{R}_{2k}^T \boldsymbol{D}_{1\mathcal{Q}_k} \boldsymbol{x}^{(k)}_{\mathcal{Q}_k,j}. \tag{11}$$

$$\boldsymbol{x}^{(k)}_{\mathcal{Q}_k,j} = (\boldsymbol{D}_{1\mathcal{Q}_k}^T \boldsymbol{D}_{1\mathcal{Q}_k})^{\dagger} \boldsymbol{D}_{1\mathcal{Q}_k}^T \boldsymbol{R}_{2k} \boldsymbol{d}_{2j}. \tag{12}$$

Pairwise Approximate K-SVD algorithm

Require: initial dictionaries $\boldsymbol{D}_1 \in \mathbb{R}^{m \times n_1}$, $\boldsymbol{D}_2 \in \mathbb{R}^{m \times n_2}$
signals set $\boldsymbol{Y}_k \in \mathbb{R}^{m \times m}$, $k = 1:N$
number of iterations $T \in \mathbb{N}$

Ensure: learned dictionaries $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$

1: **for** $t = 1$ **to** T **do**
2: Sparse coding: keeping $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ fixed, compute sparse representations $\boldsymbol{X}$ using 2D OMP
Dictionary update:
3: **for** $i = 1$ **to** n_1 **do**
4: Compute the new atom $\boldsymbol{d}_{1i}$ using (9)
5: Update $\boldsymbol{x}^{(k)}_{i,\mathcal{P}_k}$ using (10), $k = 1:N$
6: **for** $j = 1$ **to** n_2 **do**
7: Compute the new atom $\boldsymbol{d}_{2j}$ using (11)
8: Update $\boldsymbol{x}^{(k)}_{\mathcal{Q}_k,j}$ using (12), $k = 1:N$

Complexity:

- 2D OMP is $\mathcal{O}(m n_1 n_2)$, $1/m$ that of standard OMP
- dictionary update is $O(sm^2N)$, same as AK-SVD
- faster than CP-decomposition based algorithms
- considerably faster than SeDiL

References

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Denoising

σ_{noise} / PSNR	Method	lena		barbara		boat		peppers		house	
		PSNR	SSIM	PSNR	SSIM	PSNR	SSIM	PSNR	SSIM	PSNR	SSIM
5 / 34.15	1D	38.46	0.942	37.94	0.962	36.98	0.935	37.60	0.925	39.27	0.953
	2D	38.22	0.940	37.69	0.961	36.63	0.929	37.34	0.920	38.71	0.948
10 / 28.13	1D	35.30	0.907	34.21	0.931	33.47	0.878	34.63	0.876	35.62	0.901
	2D	35.21	0.905	34.10	0.930	33.42	0.875	34.56	0.873	35.43	0.897
20 / 22.11	1D	32.02	0.857	30.21	0.867	29.89	0.789	31.88	0.833	32.47	0.857
	2D	31.94	0.855	30.08	0.863	29.88	0.787	31.88	0.831	32.25	0.854
30 / 18.58	1D	29.96	0.817	27.75	0.803	27.90	0.727	30.12	0.804	30.44	0.828
	2D	29.91	0.815	27.65	0.799	27.91	0.725	30.10	0.802	30.18	0.824
50 / 14.15	1D	27.37	0.754	24.66	0.685	25.47	0.642	27.66	0.755	27.47	0.762
	2D	27.32	0.752	24.57	0.681	25.41	0.638	27.72	0.754	27.35	0.758

https://cs.unibuc.ro/~pirofti, http://acse.pub.ro https://github.com/pirofti/pair_ksvd paul@irofti.net, bogdan.dumitrescu@acse.pub.ro