Energy Blowup of Sampling-based Approximation Methods

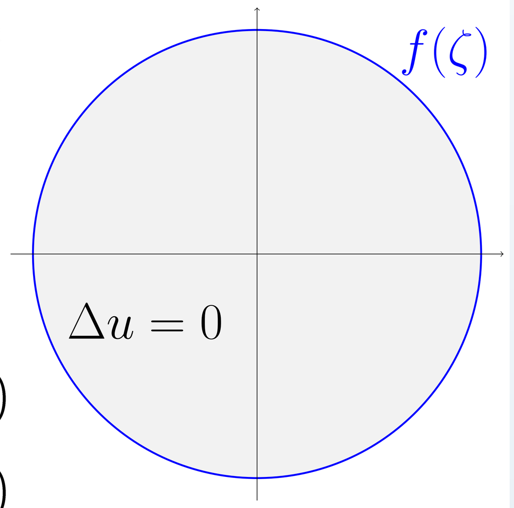
Holger Boche and Volker Pohl

Lehrstuhl für Theoretische Informationstechnik, Technische Universität München, Germany

Motivation – Dirichlet Problem on the Disk

Dirichlet problem on the unit disk

- Let $f \in \mathcal{C}(\partial\mathbb{D})$ be a given function on the unit circle
- Dirichlet problem: Find an u which is harmonic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and whose boundary values $u|_{\partial\mathbb{D}}$ coincide with the given function $f \in \mathcal{C}(\partial\mathbb{D})$.



$$(\Delta u)(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \quad \text{for all } z \in \mathbb{D} \qquad (1a)$$

$$u(\zeta) = f(\zeta) \qquad \text{for all } \zeta \in \partial\mathbb{D} \quad (1b)$$

- The unique solution is known to be the *Poisson integral of* f:

$$u(z) = (\mathrm{P}f)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\tau}) \frac{1-r^2}{1-2r\cos(\theta-\tau)+r^2}\,\mathrm{d}\tau \qquad (2)$$

for every $z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}$.

Dirichlet principle and Dirichlet energy

Any solution of the Dirichlet problem can be obtained by minimizing a particular energy functionals under the side constraint (1b):

$$u = \arg\min_{w \in \mathcal{H}_{\mathcal{D}}(\mathbb{D})} \mathrm{D}[w] \quad \text{s. t.} \quad \widehat{w}(\zeta) = f(\zeta) \quad \text{for all } \zeta \in \partial\mathbb{D}$$

wherein $\mathrm{D}[w]$ represents the Dirichlet energy of w, defined by

$$\mathrm{D}[w] = \frac{1}{2\pi}\iint_{\mathbb{D}} \|\mathrm{grad}(w)\|^2_{\mathbb{R}^2}\,\mathrm{d}A = \sum_{n\in\mathbb{Z}} |n|\,|c_n(\widehat{w})|^2 \ ,$$

and $\widehat{w}(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{r\to 1} w(r\mathrm{e}^{\mathrm{i}\theta})$ is the boundary function of w, and

$$\mathcal{H}_{\mathcal{D}}(\mathbb{D}) = \{w \in \mathcal{H}(\mathbb{D}) \text{ with } \mathrm{D}[w] < +\infty\}$$

is the set of all harmonic functions in $\mathbb{D}$ with finite Dirichlet energy.

Sampling-based Signal Processing

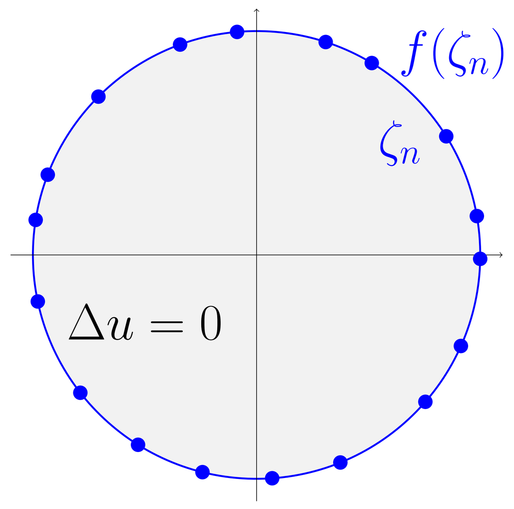
Computations on digital signal processors are always sampling-based

Sampling-based Dirichlet problem

- The Poisson integral (2) can not be calculated exactly but it can only approximated from finitely many samples $\mathcal{Z}_N = \{\zeta_n\}_{n=1}^N \subset \partial\mathbb{D}$

$$\widetilde{u}_N(z) = (\mathrm{P}_N f)(z) = \textstyle\sum_{n=1}^N f(\zeta_n)\,\mathcal{K}_N(\zeta_n, z) \ .$$

- Example: 1. Approximate f from the given samples $\{f(\zeta_n)\}$ by a polynomial or spline: $f_N = \mathrm{A}_N(f)$
  2. Apply Poisson integral: $\widetilde{u}_N = \mathrm{P}_N(f) = \mathrm{P}\mathrm{A}_N(f)$



Problem Statement

Is it possible to find a sequence of sampling-based operators $\{\mathrm{P}_N\}_{N\in\mathbb{N}}$, each concentrated on a finite sampling set $\mathcal{Z}_N = \{\zeta_n\}_{n=1}^N \subset \partial\mathbb{D}$ so that

$$\lim_{N\to\infty} \|u - \mathrm{P}_N(f)\|_\infty = 0 \quad \text{and} \quad \lim_{N\to\infty} \mathrm{D}[\mathrm{P}_N(f)] = \mathrm{D}[u] \ ?$$

Signal Spaces of Finite Energy

Seminorms measuring energy concentration

Let $f \in \mathcal{C}(\partial\mathbb{D})$ be a continuous function on the unit circle with *Fourier coefficients*

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta \ , \qquad n \in \mathbb{Z} \ .$$

For $\beta \geq 0$ define the seminorms

$$\|f\|_\beta = \sqrt{\textstyle\sum_{n\in\mathbb{Z}} |n|\,(1+\log|n|)^\beta\,|c_n(f)|^2} \ .$$

Banach spaces of finite energy

$$\mathcal{E}_\beta = \left\{f \in \mathcal{C}(\partial\mathbb{D}) \ : \ \|f\|_\beta < \infty\right\} \quad \text{with} \quad \|f\|_{\mathcal{E}_\beta} = \max\left(\|f\|_\infty, \|f\|_\beta\right) .$$

Properties

- $\|f\|^2_{\mathrm{E}} := \|f\|^2_{\beta=0}$ corresponds to the (Dirichlet) energy of f.
- $\|f\|_{\mathrm{E}} \leq \|f\|_{\beta_1} \leq \|f\|_{\beta_2}$ for all $\beta_2 > \beta_1 > 0$.
- $\mathcal{B}_{\beta_2} \subset \mathcal{B}_{\beta_1} \subset \mathcal{B}_0 \subset \mathcal{C}(\partial\mathbb{D})$ for all $\beta_2 > \beta_1 > 0$.
- Parameter β characterize the smoothness (or energy concentration) of f.
- $\mathcal{E}_\beta$ are appropriate Banach spaces for the boundary functions f of the Dirichlet problem (1). Then $\mathrm{D}(u) = \|f\|^2_{\mathrm{E}}$

Sampling-based Approximation Problem

Is it possible to find a sequence $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ of operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$, each A_N concentrated on a finite set $\mathcal{Z}_N = \{\zeta_n\}_{n=1}^N \subset \partial\mathbb{D}$ so that for all $f \in \mathcal{E}_\beta$

$$\lim\nolimits_{N\to\infty} \|f - \mathrm{A}_N(f)\|_\infty = 0 \qquad (3a)$$

$$\text{and} \quad \lim\nolimits_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathrm{E}} = \|f\|_{\mathrm{E}} \ ? \qquad (3b)$$

Remark

- It is easy to find methods $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$ such that (3a) holds for all $f \in \mathcal{E}_\beta$.
- We show: There is no method $\{\mathrm{A}_N\}_{N\in\mathbb{N}}$ which satisfies additionally (3b).

Sampling-based Approximation Methods

Definition: Let $\beta \geq 0$ be arbitrary and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be a sequence of lower semicontinuous operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$. We say that $\boldsymbol{A}$ is a *sampling-based approximation method* on $\mathcal{E}_\beta$ if it satisfies the following two properties.

(A) Concentration on finite sampling set: To every $N \in \mathbb{N}$ there exists a finite sampling set $\mathcal{Z}_N = \{\zeta_n\}_{n=1}^N \subset \partial\mathbb{D}$ such that for all $f_1, f_2 \in \mathcal{E}_\beta$

$$f_1(\zeta_n) = f_2(\zeta_n) \qquad \text{for all } \zeta_n \in \mathcal{Z}_N$$

$$\text{implies} \quad [\mathrm{A}_N(f_1)](\zeta) = [\mathrm{A}_N(f_2)](\zeta) \qquad \text{for all } \zeta \in \partial\mathbb{D} \ .$$

(B) Convergence on a dense subset: There exists a dense subset $\mathcal{M} \subset \mathcal{E}_\beta$ so that

$$\lim_{N\to\infty} \|f - \mathrm{A}_N(f)\|_{\mathcal{E}_\beta} = 0 \qquad \text{for all } f \in \mathcal{M} \ .$$

Remark

Some or even all operators $\mathrm{A}_N : \mathcal{E}_\beta \to \mathcal{E}_\beta$ might be non-linear.

Energy Blowup – Approximations

Theorem: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be a sampling-based approximation method satisfying Properties (A) and (B). Then the set

$$\mathcal{R}(\boldsymbol{A}) = \{f \in \mathcal{E}_\beta \ : \ \limsup\nolimits_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathrm{E}} = +\infty\}$$

is a residual set in $\mathcal{E}_\beta$ (i.e. its complement is a meager set).

Remark

- For any sampling-based approximation method $\boldsymbol{A}$ there always exist functions $f \in \mathcal{E}_\beta$ for which the Dirichlet energy of the approximation $\|\mathrm{A}_N(f)\|_{\mathrm{E}}$ tends to infinity as $N \to \infty$.
- This is even true for energy concentrations $0 \leq \beta \leq 1$.
- Conjecture: For $\beta \geq 1$ there always exist sampling-based approximation methods $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ which satisfy (3) for all $f \in \mathcal{E}_\beta$.

Application – Dirichlet Problem

We consider the following general algorithm for calculating the solution u of the Dirichlet problem (1) for arbitrary boundary values $f \in \mathcal{E}_\beta$ with $\beta \in [0,1]$:

1. Interpolate the given samples $\{f(\zeta_n) : n = 1, 2, \dots, N\}$ of f by an appropriate continuous function (e.g. spline or trigonometric polynomial) to obtain an approximation $f_N = \mathrm{A}_N(f)$ of f.
2. Calculate the Poisson integral (2) of f_N to obtain an approximation of the solution u:

$$\widetilde{u}_N(z) = (\mathrm{P}f_N)(z) = (\mathrm{P}\,[\mathrm{A}_N(f)])(z) \ , \quad z \in \mathbb{D} \ .$$

Energy Blowup – Dirichlet Problem

Corollary: Let $0 \leq \beta \leq 1$ be arbitrary and let $\boldsymbol{A} = \{\mathrm{A}_N\}_{N\in\mathbb{N}}$ be a sampling-based approximation method satisfying Properties (A) and (B). Then the set

$$\{f \in \mathcal{E}_\beta \ : \ \limsup\nolimits_{N\to\infty} \mathrm{D}\big[\mathrm{P}\,[\mathrm{A}_N(f)]\big] = +\infty\}$$

is a residual set in $\mathcal{E}_\beta$.

Discussion and Extensions

- We showed a fundamental limitation of digital signal processing methods.
- Similar results exist for Hilbert transform approximations.
- Dirichlet problem: Straightforward extension to arbitrary regions with a Jordan curve as boundary (Riemann mapping theorem).
- Dirichlet energy can not be calculated by a single limit $\lim_{N\to\infty} \|\mathrm{A}_N(f)\|_{\mathrm{E}}$ Do there exist sampling-bases algorithms which involve multiple limits

$$\lim_{N_1\to\infty} \dots \lim_{N_k\to\infty} \|\mathrm{A}_{N_1,\dots,N_k}(f)\|_{\mathrm{E}} \ ?$$

$\Rightarrow$ *Solvability Complexity Index* and *Towers of Algorithms*

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